Conditions for 90° Phase-Invariant Block-Coded QAM

Werner Henkel

Abstract—**Based on Zinoviev's generalized concatenated codes, conditions for the construction of 90° phase-invariant QAM are derived. Furthermore, a proposal for the necessary differential en/decoding is made. The conditions for phase invariance are specialized for the case of Reed–Muller codes as outer codes of the generalized concatenation.**

I. Introduction

FOR coded phase-shift keying, several measures have been developed to improve its behavior in the presence of phase instabilities. One possibility is to periodically insert subsets of the modulation alphabet into the sequence of coded symbols, denoted by time-variant or hybrid coded modulation [1], [2]. In the case of coded 8-PSK, this means the insertion of QPSK, doubling the hold range of the phase loop compared with 8-PSK. This first measure leads to a reduced probability of so-called cycle slips and of the resulting error bursts. Another possibility is to ensure that after a cycle slip, another stable working point is reached immediately, shortening the corresponding bursts. This is achieved by a phase invariance in the coded modulation. Such phase-invariant schemes have been proposed for coded M-PSK in [3]–[11]. Of course, the best performance is achieved if both strategies are combined.

Not too much work has been done concerning phase-invariant coded quadrature amplitude modulation (QAM). Proposals for cross QAM such as 32-QAM based on nonlinear convolutional codes can be found in [12], and recently, in [13] a solution has been presented by encoding I and Q independently with convolutional codes. This obviously is a suboptimal approach, but nevertheless noteworthy.

This contribution specifies conditions for 90° phase-invariant block-coded QAM. The conditions will be derived in two steps. First, phase invariance of the code will be ensured. This does not guarantee invariance according to the coded information symbols. This is achieved in a second step, considering the necessary differential en/decoding. The two steps will be denoted by "phase invariance" and "differential invariance." Both steps lead to conditions that each have to be fulfilled. However, the conditions will turn out not to be very stringent. Several different codes may be chosen. In the case of Reed–Muller codes, conditions for their orders are derived.

To begin with, the structure of the considered coded modulation is explained. The following sections are devoted to the phase invariance of the codes and the necessary differential coding.

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The author is with the Research Center, Deutsche Bundespost Telekom, D-64276 Darmstadt, Germany.

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II. Coded QAM Based on Zinoviev's Generalized Concatenated Codes

For simplicity, the structure of the block-encoded QAM is explained by means of the special case 16-QAM.

The set partitions of 16-QAM are given in Fig. 1. According to Zinoviev's GCC [14]–[16], the points of the 16-QAM are regarded as an *inner code.* Four binary *outer codes* are needed to encode the four partitions of 16-QAM given in Fig. 1, each of the same length n. Written in matrix form

$$\mathcal{A} \ni A = \begin{pmatrix} a^{(1)} \\ a^{(2)} \\ a^{(3)} \\ a^{(4)} \end{pmatrix} = \begin{pmatrix} a_1^{(1)}, a_2^{(1)}, \cdots, a_n^{(1)} \\ a_1^{(2)}, a_2^{(2)}, \cdots, a_n^{(2)} \\ a_1^{(3)}, a_2^{(3)}, \cdots, a_n^{(3)} \\ a_1^{(4)}, a_2^{(4)}, \cdots, a_n^{(4)} \end{pmatrix} \begin{matrix} \leftarrow \in \mathcal{A}^{(1)} \\ \leftarrow \in \mathcal{A}^{(2)} \\ \leftarrow \in \mathcal{A}^{(3)} \\ \leftarrow \in \mathcal{A}^{(4)} \end{matrix} \tag{1}$$

The rows are codewords of the outer codes $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$, $\mathcal{A}^{(3)}$, and $\mathcal{A}^{(4)}$. The columns select the corresponding points of the 16-QAM. The first component determines the 8-"QAM" subset, the second a 4-"QAM" subset of the 8-"QAM" set, the third a 2-"QAM" subset of the 4-"QAM" set, and the fourth component decides which point of the 2-"QAM" set will be taken. Fig. 1 illustrates the procedure. The minimum squared Euclidean distance of two such schemes is known to be

$$d_{E_{\min}} \geq \min_j \left\{ d_H^{(j)} \cdot d_E^{(j)} \right\} \tag{2}$$

where $d_H^{(j)}$ is the minimum Hamming distance of the jth outer code and $d_E^{(j)}$ is the minimum squared Euclidean distance between the corresponding $2^{(4-j)}$-"QAM" subsets.

The described code construction is often referred to as *multilevel coding* (see, e.g., [6], [7]). The term GCC is chosen because it is more general, and multilevel codes in the Euclidean space are a special case.

A generalization to 2^i-QAM is straightforward. The only thing that should be noted is that the numbering $(a_m^{(i)}, \cdots, a_m^{(3)})$ inside the subsets corresponding to $(a_m^{(2)}, a_m^{(1)})$ has to be chosen as rotated versions of one of the subsets (see Fig. 1).

The partitions of 32-cross QAM and the numbering for 64-QAM are given in Figs. 2 and 3, respectively.

The following section derives the necessary and sufficient conditions for 90° phase invariance.

III. Necessary and Sufficient Conditions for Phase Invariance with Respect to Multiples of $\pi/2$

The signal space code is rotationally invariant with respect to *multiples* of $\pi/2$ if and only if it is invariant with respect to a $\pi/2$ rotation. Therefore, we need to consider only the rotation by $\pi/2$.

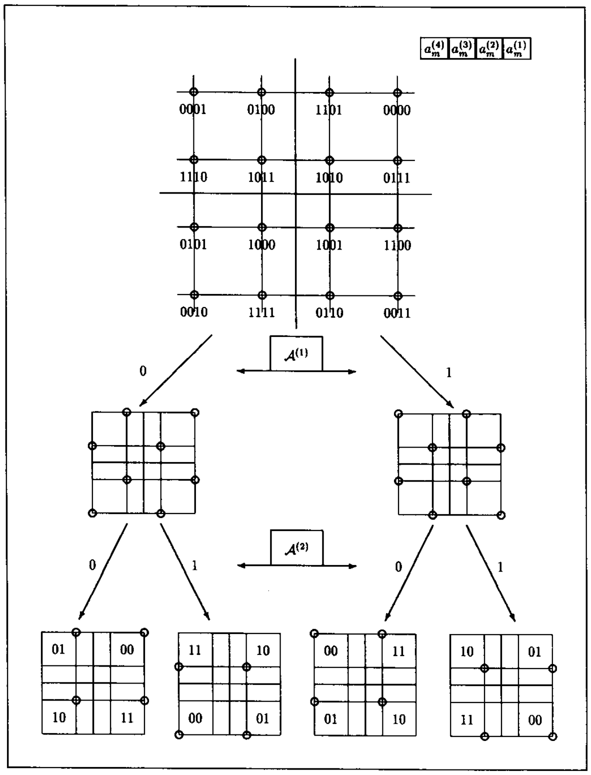

Fig. 1. Set partitions of 16-QAM.

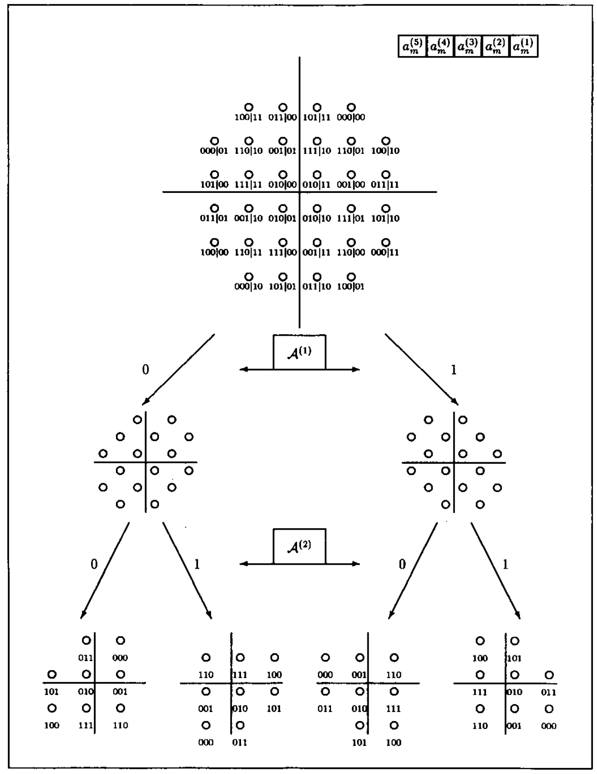

Fig. 2. Set partitions of 32-cross QAM.

For illustration, Table I shows the effect of a 90° shift in the case of 16-QAM.

Because of our particular numbering of the QAM points, only the components of $a^{(1)}$ and $a^{(2)}$ are changed according to

$$
\begin{aligned}
a_s^{(1)} &= (1, \cdots, 1) + a^{(1)} \\
a_s^{(2)} &= a^{(1)} + a^{(2)} \\
a_s^{(j)} &= a^{(j)}, \qquad j = 3, \cdots, i
\end{aligned} \tag{3}
$$

Combining the binary numbers $(a_m^{(2)}, a_m^{(1)})$ to obtain a representation of a mod-4 number, we may describe the 90° shift by an addition of

$$
\left(\underbrace{(1, \overbrace{0, \cdots, 0}^{i-2})^T, (1, 0, \cdots, 0)^T, \cdots, (1, 0, \cdots, 0)^T}_{n} \right) \operatorname{mod}(4, \overbrace{2, \cdots, 2}^{i-2})^T \tag{4}
$$

where T denotes transposition and $\operatorname{mod}(4, 2, \cdots, 2)$ stands for (mod 4, mod 2, $\cdots$, mod 2). This corresponds to Massey's ring code formulation for M-PSK [8]. Thus, to achieve 90° invariance, we demand the code construction to be invariant against the addition of the term in (4) $\operatorname{mod}(4, 2, \cdots, 2)$.

Regarding (3) and presuming linearity, we obtain the following necessary and sufficient conditions for phase invariance

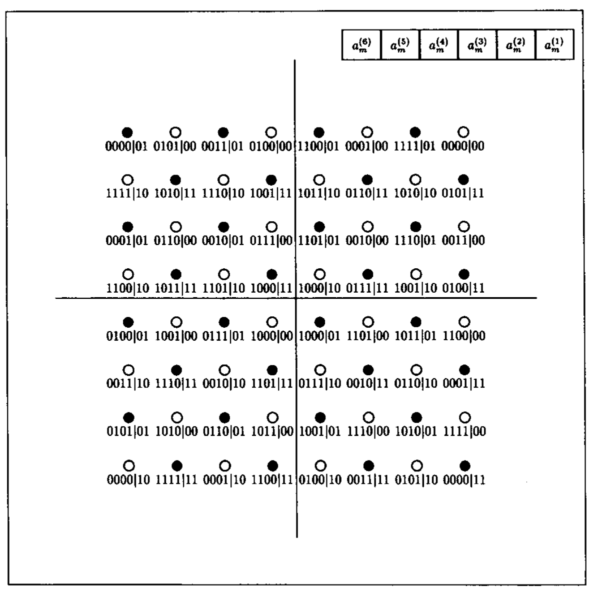

Fig. 3. Numbering according to the set partitions of 64-QAM.

with respect to multiples of $\pi/2$:

$$
\boxed{\begin{array}{c} (1, 1, \cdots, 1) \in \mathcal{A}^{(1)} \\ \mathcal{A}^{(1)} \subset \mathcal{A}^{(2)} \end{array}} \tag{5}
$$

TABLE I

+ 90°		
0 0 0 0	⟶	0 0 0 1
0 0 0 1	⟶	0 0 1 0
0 0 1 0	⟶	0 0 1 1
0 0 1 1	⟶	0 0 0 0
0 1 0 0	⟶	0 1 0 1
0 1 0 1	⟶	0 1 1 0
0 1 1 0	⟶	0 1 1 1
0 1 1 1	⟶	0 1 0 0
1 0 0 0	⟶	1 0 0 1
1 0 0 1	⟶	1 0 1 0
1 0 1 0	⟶	1 0 1 1
1 0 1 1	⟶	1 0 0 0
1 1 0 0	⟶	1 1 0 1
1 1 0 1	⟶	1 1 1 0
1 1 1 0	⟶	1 1 1 1
1 1 1 1	⟶	1 1 0 0

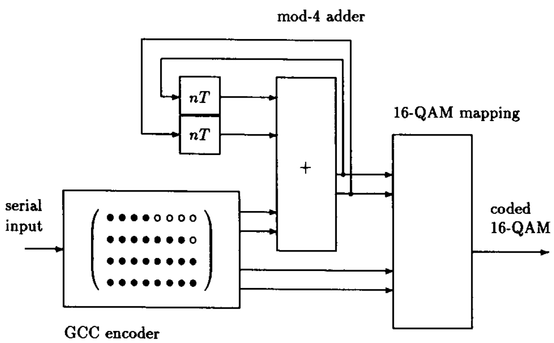

Fig. 4. Differential encoding for 90° phase-invariant block-coded 16-QAM.

With this result, only the invariance of the code against phase shifts is ensured, not the invariance of the information part itself. A differential en- and decoding is necessary. The following section describes this differential coding, and additionally, resulting conditions for the code construction.

IV. Differential Coding for 90° Phase-Invariant Block-Coded 2^i-QAM

The differential coding described here is similar to those proposed by Oerder and Meyr in [3] and the author in [10], [11]. It consists of a differential encoding mod 4 for $(a^{(2)}, a^{(1)})$ over a modulation interval of block length n after the GCC encoder (see Fig. 4). To avoid a 3 dB loss, the differential decoder has to be positioned after the GCC decoding (see Fig. 5). Thus, the GCC schemes must additionally be invariant against differential encoding mod 4.

In order to find out which additional conditions are imposed, we consider the addition of the first two GCC codewords in their mod-4 representation. We have to ensure closure with respect to addition. Then, the differential encoding yields a valid codeword, allowing the differential decoding to be performed after GCC decoding.

Let $A_1^{(1)}, A_2^{(2)} \in \mathcal{A}^{(1)}$, $A_1^{(2)}, A_2^{(2)} \in \mathcal{A}^{(2)}$. The addition of $(A_1^{(2)}, A_1^{(1)})$ and $(A_2^{(2)}, A_2^{(1)})$ mod 4 is given by

$$2 \cdot (A_1^{(2)} + A_2^{(2)}) + (A_1^{(1)} + A_2^{(1)}) \qquad (6)$$

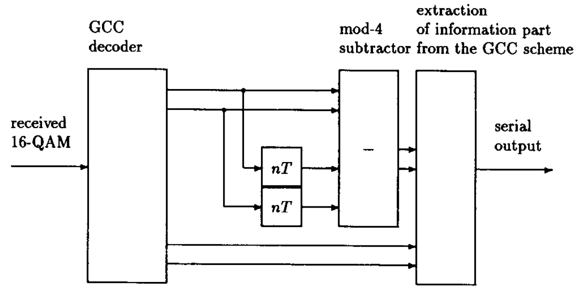

Fig. 5. Differential decoding.

TABLE II

(j)	(n,k,d_H)	RM(r,m)	d_E	$d_H \cdot d_E$	$d_H \cdot d_E/2$ /dB	$R = \Sigma k^{(j)}/\Sigma n^{(j)}$
1	(8, 4, 4)	(1,3)	1	4	3	
2	(8, 7, 2)	(2,3)	2	4	3	27/32=0.84375
3	(8, 8, 1)	(3,3)	4	4	3	> 3/4
4	(8, 8, 1)	(3,3)	8	8	(6)	
1	(16, 5, 8)	(1,4)	1	8	6	
2	(16, 11, 4)	(2,4)	2	8	6	47/64=0.734375
3	(16, 15, 2)	(3,4)	4	8	6	< 3/4
4	(16, 16, 1)	(4,4)	8	8	6	
1	(32, 16, 8)	(2,5)	1	8	6	
2	(32, 26, 4)	(3,5)	2	8	6	105/128=0.820312
3	(32, 31, 2)	(4,5)	4	8	6	> 3/4
4	(32, 32, 1)	(5,5)	8	8	6	

where $(A_k^{(2)}, A_k^{(1)})$ denotes the binary components of the mod-4 numbers. Assuming linearity, we first realize that, of course, $(A_1^{(1)} + A_2^{(1)}) \in \mathcal{A}^{(1)}$ mod 2. A carry occurs in positions where both $A_1^{(1)}$ and $A_2^{(1)}$ have ones or, equivalently, where the componentwise product $A_1^{(1)} \cdot A_2^{(1)}$ yields ones. Thus, products of codewords from $\mathcal{A}^{(1)}$ have to be in $\mathcal{A}^{(2)}$.

$$\boxed{A_1^{(1)} \in \mathcal{A}^{(1)}, \quad A_2^{(1)} \in \mathcal{A}^{(1)} \Rightarrow A_1^{(1)} \cdot A_2^{(1)} \in \mathcal{A}^{(2)}} \qquad (7)$$

If we choose the block codes in the GCC scheme to be Reed–Muller codes [17], [10], [11], the conditions for the codes turn into conditions for the orders $r^{(j)}$ of the RM codes of length $n = 2^m$:

1) Phase invariance of $\mathcal{A}$: $r^{(2)} \geq r^{(1)} \vee r^{(2)} = m$

2) Differential invariance: $r^{(2)} \geq 2 \cdot r^{(1)} \vee r^{(2)} = m$. (8)

The second condition includes the first one.

Three simple examples with RM codes yielding asymptotic coding gains of 3–6 dB may conclude these considerations. See Table II.

The overall asymptotic coding gain is given by $(d_{E_{\min}} \div 2)dB = 10 \log_{10} \min_j\{(d_H^{(j)} \cdot d_E^{(j)})/2\}$, where 2 in the denominator is the squared Euclidean distance of the uncoded 8-"QAM" in the form given in Fig. 1. This 8-"QAM" constellation has been chosen to obtain integer-valued distances. In any case, even the configuration of Fig. 6 has a squared Euclidean distance that is only by a factor of 1.0567 above that of the chosen 8-"QAM" subset (same signal energy presumed).

Rates greater than 3/4 enable us to provide additional outer codes, e.g., Reed–Solomon codes, to come to an overall code rate of 3/4. This leads to a further increase in coding

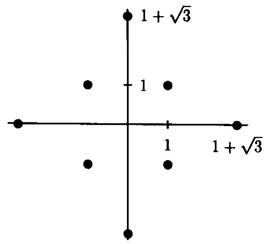

Fig. 6. 8-QAM signal set similar to CCITT V.29.

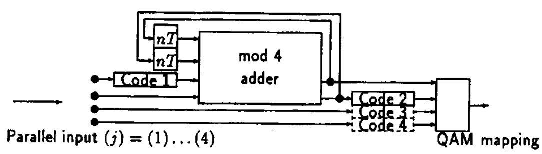

Fig. 7. Modified differential encoding for 90°-phase invariant block-coded 16-QAM.

gain. The last example of the table would allow us to use an RS code with a redundancy of 8.6%. This concatenated scheme is surely comparable with the best known nonphase-invariant modulation codes. Furthermore, it is not necessary to increase the transmission rate to compensate for the introduced redundancy, as has been proposed in [13].

One may have realized that the third example does not fulfill the second condition in (8). Indeed, another differential encoder is necessary to free us from the second condition (differential invariance). By placing the differential encoder modulo 4 between the encoder stages of the codes (1) and (2), the differential encoder operates only in the information part (assuming systematic codes) of code (2), not on the whole codeword (see Fig. 7). This means that the information part is correctly regenerated by the chain of differential encoder and decoder. However, after differential decoding, the parity symbols may not belong to a valid codeword, unless the second condition is fulfilled, too. Of course, this is no disadvantage because after decoding, parity symbols are omitted anyway.

The decoding of GCC may be performed by means of multistage soft-decision decoding, utilizing results of the trellis structure of RM codes by Forney [18]. Recent publications on similar schemes [19] show that the complexity of decoding is comparable with conventional trellis decoding.

V. Conclusions

Based on binary set partitions of 2^i-QAM, conditions for the construction of 90° phase-invariant block-coded modulation together with the necessary differential en/decoding have been specified. The corresponding ring-code description proposed by Massey for M-PSK has also been stated for QAM. Furthermore, the conditions have been applied to Reed –Muller codes, yielding simple conditions for their orders. Some examples are given that show that the coding gain achieved is comparable to that of the best known codes.

References

[1] J. Hagenauer and C.-E. W. Sundberg, "Hybrid trellis-coded 8/4–PSK modulation systems," *IEEE Trans. Commun.,* vol. 38, pp. 602–614, May 1990.
[2] M. Bertelsmeier and G. Komp, "Trellis-coded 8PSK with embedded QPSK," *IEEE J. Select. Areas Commun.,* vol. 7, Dec. 1989.
[3] M. Oerder and H. Meyr, "Rotationally invariant trellis codes for MPSK modulation," *AEÜ,* vol. 41, no. 1, pp. 28–32, 1987.
[4] L.-F. Wei, "Rotationally invariant trellis-coded modulation with multidimensional M-PSK," *IEEE J. Select. Areas Commun.,* vol. 7, Dec. 1989.
[5] S. S. Pietrobon, R. H. Deng, A. Lafanechére, G. Ungerboeck, and D. J. Costello, "Trellis-coded multidimensional phase modulation," *IEEE Trans. Inform. Theory,* vol. 36, pp. 63–89, Jan. 1990.
[6] H. Imai and S. Hirakawa, "A new multilevel coding method using error-correcting codes," *IEEE Trans. Inform. Theory,* vol. IT-23, pp. 371–377, May 1977.
[7] A. R. Calderbank, "Multilevel codes and multistage decoding," *IEEE Trans. Commun.,* vol. 37, pp. 222–229, Mar. 1989.
[8] J. L. Massey, "Ring codes for phase modulation," presented at the Kolloquium für "Elektronik und Nachrichtentechnik," ETH Zürich, Switzerland, Feb. 1990.
[9] T. Kasami, T. Takata, T. Fujiwara, and S. Lin, "On linear structure and phase rotation invariant properties of block M-PSK modulation codes," *IEEE Trans. Inform. Theory,* vol. 37, pp. 164–167, Jan. 1991.
[10] W. Henkel, "Phase-invariant coded phase shift keying using Reed–Muller codes," *AEÜ,* no. 3, pp. 125–130, 1992. (The results obtained independently in this paper are similar to those in [9]).
[11] W. Henkel, "Measures to counteract the influence of phase instabilities on coded-modulation systems," presented at ICDSC-9, Copenhagen, Denmark, May 1992.
[12] L.-F. Wei, "Rotationally invariant convolutional channel coding with expanded signal space—Part 1/2," *IEEE J. Select. Areas Commun.,* vol. SAC-2, pp. 659–686, Sept. 1984.
[13] Y. Saito, "Trellis coded modulations for multi-state QAM—Which is better, signal space expansion of bandwidth expansion?," *Trans. IEICE,* vol. E 73, pp. 1666–1672, Oct. 1990.
[14] V. A. Zinoviev, "Generalized concatenated codes for channels with error bursts and independent errors," translated from *Problem Peredachi Informatsii,* vol. 17, pp. 53–56, Oct./Dec. 1981. New York: Plenum, 1982.
[15] V. A. Zinoviev, V. V. Zyablov, and S. L. Portnoy, "Concatenated methods for construction and decoding of codes in Euclidean space," preprint, Institute for Problems of Inform. Transmission, USSR Acad. Sci., 1987.
[16] F. J. MacWilliams and N, J. A. Sloane, *The Theory of Error-Correcting Codes,* 4th ed. Amsterdam, New York, Oxford, Tokyo: North-Holland, 1977, pp. 590–592.
[17] R. E. Blahut *Theory and Practice of Error Control Codes.* Reading, MA, Menlo Park, CA, Amsterdam: Addison-Wesley, 1983.
[18] G. D. Forney, "Coset codes—Part I/II," *IEEE Trans. Inform. Theory,* vol. 34, pp. 1123–1187, Sept. 1988.
[19] Y. Be'ery, B. Shahar, and J. Snyders, "Fast decoding of the Leech lattice," *IEEE J. Select. Areas Commun.,* vol. 7, pp. 959–967, Aug. 1989.