Probability of frame synchronisation failure for binary and complex-valued sequences

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Abstract: Several different sequences for frame synchronisation have been published in the literature. Recently, even sequences using complex signal alphabets have been specified. Whether binary or complex, they are always chosen according to the properties of their autocorrelation function. However, the probability of being out of synchronisation has not as such been considered sufficiently. The paper serves as a compendium of derivations of such probability formulas for various applications and conditions, including both binary and complex signal constellations. Hard quantisation as well as the continuous case are treated.

1 Introduction

Binary sync sequences for frame synchronisation are found in nearly every data transmission system. Depending on the application, several specialised sequences have been developed. Some of them are

- *m*-sequences or PN (pseudo-noise) sequences
- Barker sequences
- Williard sequences
- Neuman–Hofman sequences
- Lindner sequences
- Bauderon–Laubie sequences

A few of them will be described below in a little more detail. The criteria for selecting such sync sequences have always been based on the autocorrelation function. This results from the maximum of the cross-correlation or its absolute value being used to synchronise, without taking into account the correction term introduced by Massey [1].

Based on the maximum of the crosscorrelation, an upper bound for the probability of being out of sync, subsequently denoted as 'sync error probability', has been derived by Maury and Style [2]. In this contribution, the exact solution is presented and a similar upper bound is derived from it. Both derivations rely on hard quantisations of the received signal. Most of this paper will be dedicated to the continuous case. AWGN (additive white Gaussian noise) will be assumed. The binary as well as a complex-valued alphabet will be handled.

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Recently, some work was done to define sync sequences over complex alphabets and especially over polyphase alphabets [3–5]. Before results of this research were published, we did some computations to search for Barker-like sequences over complex alphabets. Then we focussed on the derivation of sync-error probabilities that were not reported in the literature. Thus, this work may supplement results in the search for complex sync sequences.

In this paper, a short preliminary overview of some of the well-known binary sync sequences is presented. *m*- and Barker sequences, and the differences in the way their autocorrelations are defined, are explained. Only Barker and Barker-like sequences are examined as examples for computations. Nevertheless, the formulas obtained are also valid for other sequences. To point out some relations, the recently published papers on complex sequences are also incorporated. Derivations for sync error probabilities under several conditions are given. First, the discrete case is handled, leading to the above-mentioned upper bound as an approximation of the exact solution. The continuous case is studied for data existing outside the sync word as well as assuming the samples outside to be zero. These two possibilities are investigated for the binary alphabet and especially for complex alphabets. Fig. 1 shows the different cases to be studied.

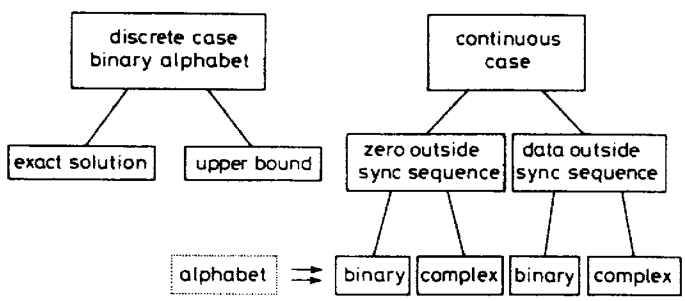

Fig. 1 *Different cases for which sync-error probabilities are derived*

2 Definition of some binary sync sequences

2.1 *m*-sequences

These sequences, also called pseudonoise sequences, have an ideal form of the autocorrelation function. The cyclic autocorrelation†

$$R(s) = \sum_{i=0}^{N-1} a_i a_{i+s}^* \bigg|_{\text{mod } N} \tag{1}$$

† The star symbolises conjugation in the complex case.

equals

$$R(s) = \begin{cases} N & s = 0 \\ -1 & s = 1, \ldots, N-1 \end{cases} \qquad (2)$$

The only drawback is that the m-sequence has to be transmitted twice to lead to the given $R(s)$, at least in an interval of width $N + 1$. Otherwise random data would be included in parts of the correlation, leading to a deviation from eqn. 2.

m-sequences follow an appealing mathematical structure. They are defined by shift registers, whose taps are given according to a primitive polynomial (see e.g. Reference 6, pp. 406–412) over the extension field GF(2^m) of the binary GF(2). The resulting length of the sequence is $2^m - 1$.

The concept of m-sequences has been generalised by Popović [5] to develop complex sequences with such ideal autocorrelation properties. His definition relies on shift registers over GF(q), which means a multilevel shift register defined by a primitive polynomial over GF(q^m).

2.2 Barker and Barker-like sequences

Barker sequences rely on a noncyclic autocorrelation function, assuming zeros outside the sequence itself. This assumption corresponds to the fact that the data mean, transmitted outside the sync sequence, equals zero if $+1$ and -1 are equally probable.

The noncyclic autocorrelation is given by

$$R_n(s) = \sum_{i=0}^{N-s-1} a_i a_{i+s}^* \qquad (3)$$

Barker sequences are defined to have sidelobes satisfying

$$|R_n(s)| \leqslant 1 \quad 1 \leqslant |s| < N \qquad (4)$$

Such sequences ... known for length N ... 2, 3, 4, 5 ...

As the real data structure is not known, the shift between the sync sequence in the receiver and the one received is restricted to the range $\{-(N-1), \ldots, +(N-1)\}$, which is a usual assumption. However, the derivation can easily be adapted to other sizes of the window within which the sync sequence is searched. Only the bounds of all products and sums in which the shift s appears as varying parameter $(-(N-1) \leqslant s \leqslant N-1)$ have to be chosen according to the new shift range.

3.1 Discrete case

Depending on the amount of the shift s (see Fig. 2), there are positions of the autocorrelation giving the product $+1$, and others yielding -1; these will be called positive or negative correlated, respectively. Outside the sync sequence, the data are assumed to be equally probable and uncorrelated with the sequence.

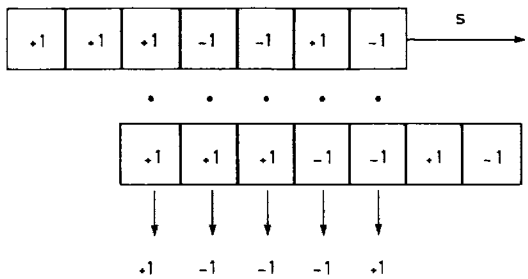

Fig. 2 *Positive and negative correlated positions of two Barker sequences of length 7 staggered by 2*

If the received sequence is erroneous with bit-error probability p, the components of the crosscorrelation have the probabilities given in Fig. 3.

tions. For simplicity, we omit the dependence on s. The total number θ of negative correlated bits are distributed to the three 'ranges' n_-, n_+ and n_u, which means that

$$\theta = \theta_+ + \theta_- + \theta_u \tag{8}$$

Then, the probability P_n of having a total of θ negative correlated bits is

$$P_n(s, \theta) = \sum_{\theta_+ = A}^{B} \binom{n_+}{\theta_+} p^{\theta_+}(1-p)^{(n_+ - \theta_+)} \times \sum_{\theta_- = C}^{D} \binom{n_-}{\theta_-} (1-p)^{\theta_-} p^{(n_- - \theta_-)} \times \binom{n_u}{\theta_u} 0.5^{\theta_u} (1-0.5)^{(n_u - \theta_u)} \tag{9}$$

In this equation, mutual dependences for the different shifts s have not been taken into account.

The limits of the summations are determined by some simple considerations. D cannot be greater than n_- ($\theta_- \leqslant n_-$) and it cannot be greater than $\theta - \theta_+$, the number of remaining negative correlated bits, when θ_+ have already been taken from θ. Thus, D is given by

$$D = \min(n_-, \theta - \theta_+)$$

B is derived in a similar way. B cannot exceed n_+ and the total number of negative correlated positions θ:

$$B = \min(n_+, \theta)$$

Likewise, the lower limit C cannot be smaller than zero and has to be greater or equal to $\theta - n_u - \theta_+$:

$$C = \max(0, \theta - n_u - \theta_+)$$

and A is also lower-bounded by zero and $\theta - n_u - n_-$:

$$A = \max(0, \theta - n_u - n_-)$$

Together with eqn. 8, the probability of having θ negative correlated positions is given by

$$P_n(s, \theta) = \sum_{\theta_+ = A}^{B} \binom{n_+}{\theta_+} p^{\theta_+}(1-p)^{n_+ - \theta_+} \times \sum_{\theta_- = C}^{D} \binom{n_-}{\theta_-} (1-p)^{\theta_-} p^{(n_- - \theta_-)} \times \binom{n_u}{\theta - \theta_+ - \theta_-} 0.5^{n_u} \tag{10}$$

$B = \min(n_+, \theta)$	$D = \min(n_-, \theta - \theta_+)$
$A = \max(0, \theta - n_u - n_-)$	$C = \max(0, \theta - n_u - \theta_+)$

The crosscorrelation function

$$R_c(s) = \sum_{i=0}^{N-1} a_i b_{i+s} \tag{11}$$

itself depends on θ according to the relation

$$R_c(s, \theta) = N - 2\theta \quad \theta \in [0, N] \tag{12}$$

Thus, the probability of negative correlated locations is also the probability of having a certain crosscorrelation function.

Now we are prepared to derive the sync-error probability for the discrete binary case. First, the exact solution is given followed by the above-mentioned upper bound resulting from an approximation.

3.1.1 *Exact solution of the sync-error probability:*

Let P_f be the sync-error probability, the probability that the crosscorrelation at an arbitrary shift $s \neq 0$ is greater or equal to that at $s = 0$. Furthermore, let P_r be the probability of being in sync

$$P_r = 1 - P_f \tag{13}$$

If $P(R_c(s = 0))$ denotes the probability of having a certain crosscorrelation at $s = 0$ and we bear in mind that to be in sync, every crosscorrelation for $s \neq 0$ has to be less than that at $s = 0$, we obtain the relation for P_r:

$$P_r = \sum_{R_c(s=0)} \left(\prod_{\substack{s=-(N-1) \\ s \neq 0}}^{N-1} P(R_c(s) < R_c(s=0)) \right) \times P(R_c(s=0)) \tag{14}$$

where $\sum_{R_c(s=0)}$ denotes the sum over all possible results of the crosscorrelation at $s = 0$. Considering eqn. 12, eqn. 14 can be rewritten as

$$P_r = \sum_{\theta_0 = 0}^{N} \left(\prod_{\substack{s=-(N-1) \\ s \neq 0}}^{N-1} \sum_{\theta_s = \theta_0 + 1}^{N} P_n(s, \theta_s) \right) P_n(0, \theta_0) \tag{15}$$

Thus, the sync-error probability P_f is

$$P_f = 1 - \sum_{\theta_0 = 0}^{N-1} \left[\prod_{\substack{s=-(N-1) \\ s \neq 0}}^{N-1} \left(\sum_{\theta_s = \theta_0 + 1}^{N} P_n(s, \theta_s) \right) \right] P_n(0, \theta_0) \tag{16}$$

Utilising the symmetry of $P_n(-s, \theta) = P_n(s, \theta)$, eqn. 16 yields

$$P_f = 1 - \sum_{\theta_0 = 0}^{N-1} \left[\prod_{s=1}^{N-1} \left(\sum_{\theta_s = \theta_0 + 1}^{N} P_n(s, \theta_s) \right) \right]^2 P_n(0, \theta_0) \tag{17}$$

The next section gives an approximation that is close to the exact solution for small bit-error probabilities.

3.1.2 *Upper bound of the sync-~~order~~ error probability:*

Since

$$\sum_{\theta_s = \theta_0 + 1}^{N} P_n(s, \theta_s) = 1 - \sum_{\theta_s = 0}^{\theta_0} P_n(s, \theta_s) \tag{18}$$

eqn. 16 can be approximated by

$$P_f \leqslant 1 - \sum_{\theta_0 = 0}^{\theta = N} P_n(0, \theta_0) \left(1 - \sum_{\substack{s=-(N-1) \\ s \neq 0}}^{N-1} \sum_{\theta_s = 0}^{\theta_0} P_n(0, \theta_s) \right) = \sum_{\substack{s=-(N-1) \\ s \neq 0}}^{N-1} \sum_{\theta_0 = 0}^{N} P_n(0, \theta_0) \sum_{\theta_s = 0}^{\theta_0} P_n(s, \theta_s) \tag{19}$$

Again, because of the symmetry of $P_n(-s, \theta) = P_n(s, \theta)$, eqn. 19 can be further simplified:

$$P_f \leqslant 2 \sum_{s=1}^{N-1} \sum_{\theta_0 = 0}^{N} P_n(0, \theta_0) \sum_{\theta_s = 0}^{\theta_0} P_n(s, \theta_s) \tag{20}$$

In Fig. 4 the exact solution is compared to the upper bound. All Barker sequences and three proposed by Bauderon and Laubie have been included. P_f has been shown to depend on the standard deviation of Gaussian noise ($p = (1/2)\,\mathrm{erfc}\,(1/\sqrt{(2)}\sigma)$), to enable a comparison with the results given below. Both solutions are seen to converge for small bit-error probabilities (small σ).

Limits for $\sigma \to \infty$ are derived in Appendix 6.1.

In the following Section, the continuous case is considered, first restricting the symbols outside the sync sequence to be sent as zeros. The more difficult situation, when data are transmitted outside, is studied afterwards.

3.2 *Continuous case*

Again, we consider the channel noise to be AWGN (additive white Gaussian noise) but, of course, with some effort, other density functions could be treated in the same way.

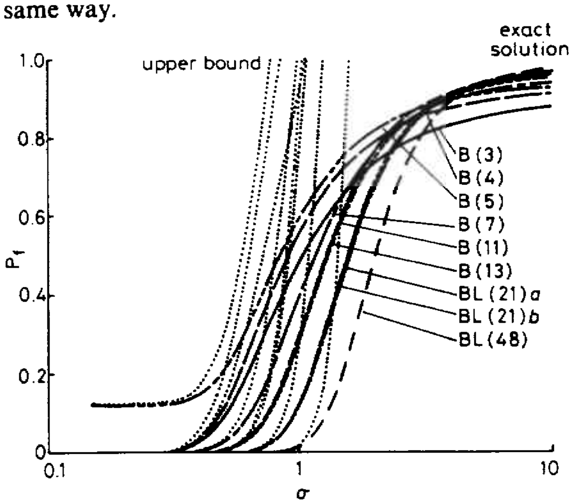

Fig. 4 *Sync-error probability in the discrete case — a comparison between upper bound and exact solution*

The considered sequences of lengths 3 to 13 are by Barker (B) and those of lengths 21 (two different sequences) and 48 are by Bauderon and Laubie (BL)

First, let the alphabet be binary with zeros transmitted outside the sync sequence.

3.2.1 Zeros outside, binary alphabet: The probability density function of one component of the crosscorrelation $(b_i a_{i+s})$ is given by

$$f(u) = \frac{1}{\sqrt{(2\pi)}\sigma} \exp - \frac{(u-\mu)^2}{2\sigma^2} \tag{21}$$

where μ = mean and σ = standard deviation. Under the condition of statistical independence, the sum of those terms formed by the crosscorrelation is again distributed normally.

Thus, $R = \sum b_i a_{i+s}$ is Gaussian with mean $\mu_s = n_+ - n_- = R_0(s)$ (autocorrelation), $\mu_{s=0} = N$, and variance $\sigma_N^2 = N\sigma^2$

$$f_s(u) = \frac{1}{\sqrt{(2\pi N)}\sigma} \exp - \frac{(u-\mu_s)^2}{2N\sigma^2} \tag{22}$$

Let $f_{s=0}(u)$ be the probability density function for $s = 0$ and $F_s(u)$ the corresponding distribution function to f_s. Then, in analogy with eqn. 15, P_r is given by

$$P_r = \int_u \left(\prod_{\substack{s=-(N-1)\\ s\neq 0}}^{N-1} F_s(u) \right) f_{s=0}(u)\, du \tag{23}$$

$$F_s(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{(2\pi N)}\sigma} \exp\left[-\frac{(x-\mu_s)^2}{2N\sigma^2} \right] dx \Bigg|_{\mu_s = n_+ - n_-}$$

Compare also eqn. 17. Some trivial mathematical operations lead to

$$P_f = 1 - \int_{u=-\infty}^{+\infty} \left[\prod_{\substack{s=-(N-1)\\ s\neq 0}}^{N-1} \frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{u-\mu_s}{\sqrt{(2N)}\sigma}\right]\right) \right] \times \frac{1}{\sqrt{(2\pi N)}\sigma} \exp\left[-\frac{(u-N)^2}{2N\sigma^2} \right] du \tag{24}$$

Because of the symmetry of the autocorrelation function, represented by $\mu_s = R_0(s)$, eqn. 24 can be written as

$$P_f = 1 - \int_{u=-\infty}^{+\infty} \left[\prod_{s=1}^{N-1} \frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{u-\mu_s}{\sqrt{(2N)}\sigma}\right]\right) \right]^2 \times \frac{1}{\sqrt{(2\pi N)}\sigma} \exp\left[-\frac{(u-N)^2}{2N\sigma^2} \right] du \tag{25}$$

Results of some computations for Barker and Barker-like sequences are illustrated in Fig. 5.

Limits for $\sigma \to \infty$ are derived in Appendix 6.2.

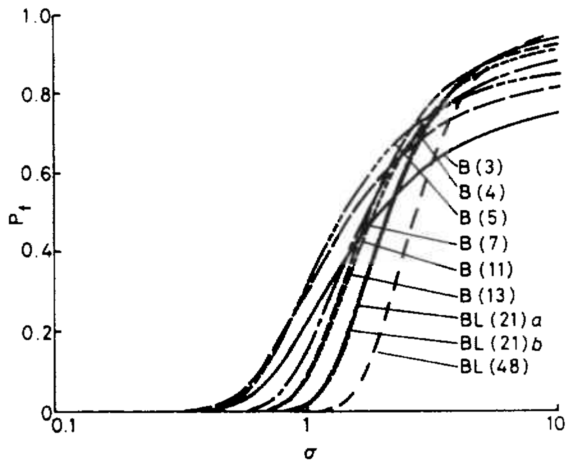

Fig. 5 *Sync-error probability in the continuous case with binary alphabet and zeros assumed outside the sequences*

3.2.2 Zeros outside, complex-valued alphabet: The considerations in this Section are based on the assumption that the real part of the complex crosscorrelation or its maximum, respectively, is used for synchronisation

$$\max_s \operatorname{Re} \left\{ \sum_{i=0}^{N-1} b_i a_{i+s}^* \right\} \tag{26}$$

Now, the random variable u is complex

$$\underline{u} = x + jy \tag{27}$$

The integral in eqn. 25 is changed into a two-dimensional integral. However, since only the real part is of any interest, the relation for P_f in eqn. 25 is only slightly modified

$$P_f = 1 - \int_{\underline{u}} \left[\prod_{s=1}^{N-1} \frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{\operatorname{Re}\{\underline{u}\} - \operatorname{Re}\{\underline{\mu}_s\}}{\sqrt{(2N)}\sigma}\right]\right) \right]^2 \times \frac{1}{2\pi N\sigma^2} \exp\left[-\frac{|\underline{u}-N|^2}{2N\sigma^2} \right] d\underline{u}, \quad \underline{\mu}_s = \underline{R}(s) \tag{28}$$

$$P_f = 1 - \frac{1}{2\pi N\sigma^2} \int_x \left[\prod_{s=1}^{N-1} \frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{x - \operatorname{Re}\{\underline{\mu}_s\}}{\sqrt{(2N)}\sigma}\right]\right) \right]^2 \times \exp\left[-\frac{(x-N)^2}{2N\sigma^2} dx \times \int_{y=-\infty}^{+\infty} \exp\left[-\frac{|y|^2}{2N\sigma^2} \right] dy \tag{29}$$

The integral over the imaginary part equals

$$\int_{-\infty}^{+\infty} \exp\left[-\frac{|y|^2}{2N\sigma^2} \right] dy = \sqrt{(2N\pi)}\sigma \tag{30}$$

This yields

$$P_f = 1 - \frac{1}{\sqrt{(2\pi N)}\sigma} \times \int_{-\infty}^{+\infty} \left[\prod_{s=1}^{N-1} \frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{x - \operatorname{Re}\{\underline{\mu}_s\}}{\sqrt{(2N)}\sigma}\right]\right) \right]^2 \times \exp\left[-\frac{(x-N)^2}{2N\sigma^2} \right] dx \tag{31}$$

Up to now, in the continuous case, zeros outside the transmitted sync sequence have been assumed. Random data outside the sequence will entail more effort.

3.2.3 Random data outside, binary alphabet: The probability density function of a term of the crosscorrelation $(b_i a_{i+s})$ is no longer Gaussian, but a superposition of two or more (for the complex alphabet) normal densities. Assuming that in the binary case $+1$ and -1 are equally probable, the probability density function f_a of the received samples outside the overlap range of the received and receiver sync sequences are given by (see also Fig. 6)

$$f_a(u) = \frac{1}{2}\frac{1}{\sqrt{(2\pi)}\sigma}\left(\exp\left[-\frac{(u-1)^2}{2\sigma^2}\right] + \exp\left[-\frac{(u+1)^2}{2\sigma^2}\right]\right) \tag{32}$$

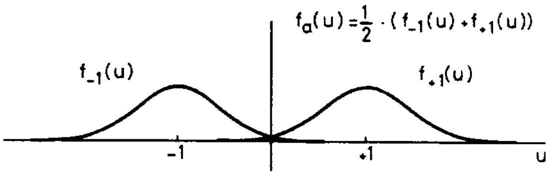

Fig. 6 *Probability density function of the received samples outside the overlap range*
Equiprobable binary data with Gaussian noise

The overlap range itself remains Gaussian with mean $\mu_s = R_0(s) = n_+ - n_-$ and variance $\sigma_N^2 = (N - |s|)\sigma^2$. Thus, the density function of that part of the crosscorrelation function representing the overlap range is

$$f_s(u) = \frac{1}{\sqrt{(2\pi)}\sqrt{(N-|s|)}\sigma}\exp\left[-\frac{(u-\mu_s)^2}{2(N-|s|)\sigma^2}\right] \tag{33}$$

The remaining components outside the overlap region have to be added. A sum of random variables corresponds to the convolution of its density functions.

$$f_{u_1+u_2}(u) = \int_{-\infty}^{\infty} f_{u_1}(\zeta) f_{u_2}(u-\zeta)\, d\zeta \tag{34}$$

The total probability density function of the crosscorrelation has now to be calculated according to

$$f_{s,tot}(u) = f_s(u) * \underbrace{f_a(u) * f_a(u) * \cdots * f_a(u)}_{s \text{ times}} \tag{35}$$

The convolution again corresponds to the multiplication of the Fourier transforms, the characteristic functions.

These are*

$$f_a(u) \circ\!\!-\!\!\bullet\, \Phi_a(\omega) = \tfrac{1}{2}\exp\left[-\left(\frac{\sigma^2\omega^2}{2}\right)\right] \times (\exp[-j\omega] + \exp[+j\omega]) \tag{36}$$

$$f_s(u) \circ\!\!-\!\!\bullet\, \Phi_s(\omega) = \exp[-j\omega\mu_s] \times \exp\left[-\frac{(N-|s|)\sigma^2\omega^2}{2}\right] \tag{37}$$

The Fourier transform of eqn. 35, together with eqns. 36, and 37, results in

$$f_{s,tot}(u) \circ\!\!-\!\!\bullet\, \Phi_{s,tot}$$

$$= \Phi_s(\omega)(\tfrac{1}{2})^{|s|}\exp\left[-\frac{|s|\sigma^2\omega^2}{2}\right] \times (\exp[-j\omega] + \exp[+j\omega])^{|s|} \tag{38}$$

$$\Phi_{s,tot} = (\tfrac{1}{2})^{|s|}\exp\left[-\frac{N\sigma^2\omega^2}{2}\right]\exp(-j\mu_s\omega) \times \underbrace{(\exp(-j\omega) + \exp(+j\omega))^{|s|}}_{\mathscr{X}} \tag{39}$$

where

$$\mathscr{X} = \sum_{v=0}^{|s|}\binom{|s|}{v}\exp[-j\omega(|s|-v)]\exp(+j\omega v)$$

$$= \sum_{v=0}^{|s|}\binom{|s|}{v}\exp[-j\omega(|s|-2v)]$$

$$\Rightarrow f_{s,tot}(u) \circ\!\!-\!\!\bullet\, \Phi_{s,tot}$$

$$= (\tfrac{1}{2})^{|s|}\exp\left(-\frac{N\sigma^2\omega^2}{2}\right)\sum_{v=0}^{|s|}\binom{|s|}{v} \times \exp\left[-j\omega\underbrace{(|s| - 2v + \mu_s)}_{\mu(s,v)}\right] \tag{40}$$

The inverse Fourier transform yields

$$\Phi_{s,tot} \bullet\!\!-\!\!\circ\, f_{s,tot}(u) = \left(\frac{1}{2}\right)^{|s|}\frac{1}{\sqrt{(2\pi N)}\sigma}\sum_{v=0}^{|s|}\binom{|s|}{v} \times \exp\left[-\frac{(u-\mu(s,v))^2}{2N\sigma^2}\right] \tag{41}$$

In order to apply eqn. 23, the distribution function

$$F_s(u) = \int_{-\infty}^{u} f_{s,tot}(x)\, dx$$

has to be calculated

$$F_s(u) = \int_{-\infty}^{u}\frac{1}{\sqrt{(2N\pi)}\sigma}\left(\frac{1}{2}\right)^{|s|}\sum_{v=0}^{|s|}\binom{|s|}{v} \times \exp\left[-\frac{(x-\mu(s,v))^2}{2N\sigma^2}\right]dx$$

$$= \left(\frac{1}{2}\right)^{|s|}\sum_{v=0}^{|s|}\binom{|s|}{v}\frac{1}{2}\left(1 + \operatorname{erf}\left[\frac{u-\mu(s,v)}{\sqrt{(2N)}\sigma}\right]\right) \tag{42}$$

Together with $f_{s=0,tot}(u)$ inserted into eqn. 23, we obtain

$$P_f = 1 - \int_{u=-\infty}^{\infty}\underbrace{\left[\prod_{s=1}^{N-1}\left(\frac{1}{2}\right)^{|s|}\sum_{v=0}^{|s|}\binom{|s|}{v}\frac{1}{2}\left(1+\operatorname{erf}\left[\frac{u-\mu(s,v)}{\sqrt{(2N)}\sigma}\right]\right)\right]^2}_{F_s(u)}\underbrace{\frac{1}{\sqrt{(2N\pi)}\sigma}\exp\left[-\frac{(u-N)^2}{2N\sigma^2}\right]}_{f_{s=0}(u)}du$$

$$\mu(s,v) = (|s| - 2v + n_+ - n_-) \tag{43}$$

The limit for $\sigma \to \infty$ is given in Appendix 6.3.

One case remains to be considered — a complex alphabet and random data outside the overlap range. This will only be studied for one special configuration, the 4-PSK signal format. Other signal sets can be handled following the same procedure, except that the number of terms in the formulas increase.

* $\circ\!\!-\!\!\bullet$ denotes Fourier transform.

3.2.4 Random data outside, complex-valued alphabet (4-PSK): The data transmitted outside the sync sequence is assumed to be equally probable, which means that all four points of the 4-PSK ($+1$, -1, $+j$, $-j$) appear with a probability of 1/4. In the presence of additive white Gaussian noise, the probability density function outside the overlap region is given by

$$f_a(\underline{u}) = \frac{1}{4}\frac{1}{2\pi\sigma^2}\left(\exp\left[-\frac{|\underline{u}-1|^2}{2\sigma^2}\right] + \exp\left[-\frac{|\underline{u}+1|^2}{2\sigma^2}\right] + \exp\left[-\frac{|\underline{u}-j|^2}{2\sigma^2}\right] + \exp\left[-\frac{|\underline{u}+j|^2}{2\sigma^2}\right]\right) \quad (44)$$

For the overlap range we again have

$$\sigma_N^2 = (N - |s|)\sigma^2 \quad (45)$$

$$\begin{aligned} f_s(\underline{u}) &= \frac{1}{2\pi(N-|s|)\sigma^2}\exp\left[-\frac{|\underline{u}-\underline{R}_0(s)|^2}{2(N-|s|)\sigma^2}\right] \\ &= \frac{1}{2\pi(N-|s|)\sigma^2}\exp\left[-\frac{|x-\mathrm{Re}\{\underline{R}_0(s)\}|^2}{2(N-|s|)\sigma^2}\right] \\ &\quad \times \exp\left[-\frac{|y-\mathrm{Im}\{\underline{R}_0(s)\}|^2}{2(N-|s|)\sigma^2}\right] \end{aligned} \quad (46)$$

where $\underline{R}_0(s)$ is the noncyclic autocorrelation given by $\underline{R}_0(s) = \sum_{i=0}^{N-s} a_i a_{i+s}^*$ (compare also eqn. 33).

In the sequel, Re $\{\underline{R}_0(s)\}$ and Im $\{\underline{R}_0(s)\}$ will be abbreviated by R and I, respectively. As in the section before, the Fourier transform of the density functions is needed. This time, of course, a two-dimensional transform has to be applied. The characteristic functions are then given by (see e.g. Reference 9)

$$\begin{aligned} f_a(\underline{u}) &\circ\!\!-\!\!\bullet\ \Phi_a(\omega_1, \omega_2) \\ &= \frac{1}{4}\left(\exp\left[-\frac{\sigma^2(\omega_1^2+\omega_2^2)}{2}\right]\exp(-j\omega_1) + \exp\left[-\frac{\sigma^2(\omega_1^2+\omega_2^2)}{2}\right]\exp(+j\omega_1) \right.\\ &\quad \left. + \exp\left[-\frac{\sigma^2(\omega_1^2+\omega_2^2)}{2}\right]\exp(-j\omega_2) + \exp\left[-\frac{\sigma^2(\omega_1^2+\omega_2^2)}{2}\right]\exp(+j\omega_2)\right) \end{aligned} \quad (47)$$

$$\begin{aligned} f_s(\underline{u}) &\circ\!\!-\!\!\bullet\ \Phi_s(\omega_1, \omega_2) \\ &= \exp\left[-\frac{\sigma^2(N-|s|)(\omega_1^2+\omega_2^2)}{2}\right] \\ &\quad \times \exp(-j\omega_1 \mathrm{R})\exp(-j\omega_2 \mathrm{I}) \end{aligned} \quad (48)$$

The total probability density function is again (see eqn. 35) determined by a convolution, this time in two variables:

$$f_{s,tot}(\underline{u}) = f_s(\underline{u}) * \underbrace{f_a(\underline{u}) * \cdots * f_a(\underline{u})}_{s \text{ times}} \quad (49)$$

This means that the corresponding two-dimensional Fourier transforms are to be multiplied

$$\Phi_{s,tot}(\omega_1, \omega_2) = \Phi_s(\omega_1, \omega_2)(\Phi_a(\omega_1, \omega_2))^{|s|} \quad (50)$$

Inserting eqns. 47 and 48 into eqn. 50 yields

$$\begin{aligned} f_{s,tot}(\underline{u}) &\circ\!\!-\!\!\bullet\ \Phi_{s,tot}(\omega_1, \omega_2) \\ &= \exp\left[-\frac{\sigma^2(N-|s|)(\omega_1^2+\omega_2^2)}{2}\right] \\ &\quad \times \exp(-j\omega_1 \mathrm{R})\exp(-j\omega_2 \mathrm{I}) \\ &\quad \times \left(\frac{1}{4}\exp\left[-\frac{\sigma^2(\omega_1^2+\omega_2^2)}{2}\right] \times (\exp(-j\omega_1) + \exp(+j\omega_1) + \exp(-j\omega_2) + \exp(+j\omega_2))\right)^{|s|} \end{aligned} \quad (51)$$

Considering that

$$(a+b+c+d)^{|s|} = \sum_{k_1+k_2+k_3+k_4=|s|} \binom{|s|}{k_1, k_2, k_3, k_4} a^{k_1} b^{k_2} c^{k_3} d^{k_4} \quad (52)$$

eqn. 51 is rewritten as

$$\begin{aligned} f_{s,tot}(\underline{u}) &\circ\!\!-\!\!\bullet\ \Phi_{s,tot}(\omega_1, \omega_2) \\ &= \left(\frac{1}{4}\right)^{|s|}\exp\left[-\frac{\sigma^2(N-|s|)(\omega_1^2+\omega_2^2)}{2}\right] \\ &\quad \times \exp(-j\omega_1 \mathrm{R})\exp(-j\omega_2 \mathrm{I}) \\ &\quad \times \sum_{k_1..=|s|}\binom{|s|}{k_1, \ldots, k_4}\exp\left[-\frac{\sigma(\omega_1^2+\omega_2^2)}{2}|s|\right] \\ &\quad \times \exp[-j\omega_1(k_1-k_2)] \\ &\quad \times \exp[-j\omega_2(k_3-k_4)] \end{aligned} \quad (53)$$

$$\begin{aligned} &= \left(\frac{1}{4}\right)^{|s|}\sum_{k_1..=|s|}\binom{|s|}{k_1, \ldots, k_4} \\ &\quad \times \exp\left[-\frac{\sigma^2 N(\omega_1^2+\omega_2^2)}{2}\right] \\ &\quad \times \exp[-j\omega_1(k_1-k_2+\mathrm{R})] \\ &\quad \times \exp[-j\omega_2(k_3-k_4+\mathrm{I})] \end{aligned} \quad (54)$$

Applying the inverse Fourier transform, $f_{s,tot}(\underline{u})$ results in

$$\begin{aligned} f_{s,tot} &= \frac{1}{2\pi N\sigma^2}\left(\frac{1}{4}\right)^{|s|}\sum_{k_1..=|s|}\binom{|s|}{k_1, \ldots, k_4} \\ &\quad \times \exp\left[-\frac{(x-k_1+k_2-\mathrm{R})^2}{2N\sigma^2}\right] \\ &\quad \times \exp\left[-\frac{(y-k_3+k_4-\mathrm{I})^2}{2N\sigma^2}\right] \end{aligned} \quad (55)$$

In order to apply eqn. 23, the distribution function

$$F_s(x_0) = \int_{x=-\infty}^{x_0}\int_{y=-\infty}^{\infty} f_{s,tot}(\underline{u} = x + jy)\,dy\,dx$$

has to be determined

$$\begin{aligned} F_s(x_0) &= \frac{1}{2\pi N\sigma^2}\left(\frac{1}{4}\right)^{|s|}\sum_{k_1..=|s|}\binom{|s|}{k_1, \ldots, k_4} \\ &\quad \times \int_{x=-\infty}^{x_0}\exp\left[-\frac{(x-k_1+k_2-\mathrm{R})^2}{2N\sigma^2}\right]dx \\ &\quad \times \underbrace{\int_{y=-\infty}^{+\infty}\exp\left[-\frac{(y-k_3+k_4-\mathrm{I})^2}{2N\sigma^2}\right]dy}_{\sqrt{(2\pi N)}\sigma} \end{aligned} \quad (56)$$

In this way, we obtain the desired sync-error probability

$$P_f = 1 - \int_{x=-\infty}^{\infty} \left[\prod_{s=1}^{N-1} \left(\frac{1}{4}\right)^{|s|} \sum_{k_1 .. = |s|} \binom{|s|}{k_1, \ldots, k_4} \frac{1}{2} \times \left(1 + \operatorname{erf}\left[\frac{x - k_1 + k_2 - \operatorname{Re}\{\underline{R}_0(s)\}}{\sqrt{(2N)}\sigma}\right]\right)\right]^2 \times \frac{1}{\sqrt{(2\pi N)}\sigma} \exp\left[-\frac{(x-N)^2}{2N\sigma^2}\right] \quad (57)$$

4 Conclusions

The probability of being out of sync has been determined under various conditions. Thus, to a certain extent, this contribution can be seen as a compendium of such relations for sync-error probabilities.

The discrete, quantised and continuous cases are examined. For the discrete (binary) case, the exact solution and an upper bound are given. In the continuous case, two different assumptions were studied — having zero or random data outside the sync word. For each assumption the cases of having a binary or a complex signal alphabet were considered. The only restriction that has been made is that the channel should be AWGN (additive white Gaussian noise). But, with some effort, it should also be possible to modify the given derivations to match other noise conditions. However, the discrete case can always be applied and yields at least an upper bound of the out-of-sync probability for the continuous case, no matter which probability density function underlies. In most cases, this is sufficient for choosing an adequate sequence for a special application. In Reference 17, such upper bounds have been derived for a special frame synchronisation developed for time-variant coded modulation. An exact computation for the continuous case was not feasible. Nevertheless, the discrete computation yielded an acceptable approximation.

5 References

1 MASSEY, J.L.: 'Optimum frame synchronisation', *IEEE Trans.*, 1972, **COM-20**, (2), pp. 115–119

2 MAURY, J.L., and STYLE, F.J.: 'Development of optimum frame synchronisation codes for Goddard space flight center PCM telemetry standards'. Los Angeles, Proc. National Telemetering Conference, June 1964

3 BÖMER, L., and ANTWEILER, M.: 'Polyphase Barker sequences', *Electron. Lett.*, 1989, **25**, (23), pp. 1577–1579

4 ZHANG, N., and GOLOMB, S.W.: 'Sixty-phase generalized Barker sequences', *IEEE Trans.*, 1989, **IT-35**, (4), pp. 911–912

5 POPOVIĆ, B.M.: 'New class of complex sequences with ideal autocorrelation', *AEÜ*, 1989, **43**, (1), pp. 13–15

6 MACWILLIAMS, F.J., and SLOANE, N.J.A.: 'The theory of error-correcting codes' (North-Holland, 1986)

7 BAUDERON, M., and LAUBIE, F.: 'Searching for weakly autocorrelated binary sequences'. Université de Bordeaux, No. I-8719, April 1987

8 NEUMAN, F., and HOFMAN, L.: 'New pulse sequence with desirable correlation properties'. Proc. Nat. Telemetry Conf., Washington DC, April 1971, pp. 272–282

9 FÖLLINGER, O.: 'Laplace- und Fourier-Transformationen' (Elitera-Verlag, Berlin, 1977)

10 SPILKER, J.J.: 'Digital communications by satellite' (Prentice-Hall, 1977)

11 TURYN, R.: 'On Barker codes of even length', *Proc. IEEE*, September 1963, **51**, p. 1256

12 LUENBERGER, D.G.: 'On Barker codes of even length', *Proc. IEEE*, January 1963, **51**, pp. 230–231

13 PAPOULIS, A.: 'Probability, random variables and stochastic processes' (McGraw-Hill, London, 1985)

14 PROAKIS, J.G.: 'Digital communications' (McGraw-Hill, London, 1985, 2nd edn.)

15 BARKER, R.H.: 'Group synchronizing of binary digital systems', *in* JACKSON, W. (Ed.): 'Communication theory' (Butterworth, 1952)

16 LINDNER, J.: 'Binary sequences up to length 40 with best possible autocorrelation function', *Electron. Lett.*, November 1975, p. 507

17 HENKEL, W.: 'Combined coding, modulation, carrier and frame synchronization on the basis of time-variant coded 8/4-PSK'. 6th Int. Conf. on Digital Processing of Signals in Communications, Loughborough, UK, 2–6 September 1991

6 Appendix

In this Appendix limits for $\sigma \to \infty$ are determined.

6.1 Limit for eqn. 14 as $\sigma \to \infty$

$$\lim_{\sigma\to\infty} P(R_c(s) < R_c(s=0)) = \frac{1}{2}$$

$$\Rightarrow P_r = \sum_{R_c(s=0)} \left(\prod_{\substack{s=-(N-1) \\ s\neq 0}}^{N-1} \frac{1}{2} \right) P(R_c(s=0))$$

$$= \left(\frac{1}{2}\right)^{2(N-1)}$$

$$\Rightarrow P_f = 1 - \left(\frac{1}{2}\right)^{2(N-1)}$$

6.2 Limit for eqn. 25 as $\sigma \to \infty$

$$\lim_{\sigma\to\infty} \left(1 + \operatorname{erf}\left[\frac{u-\mu_s}{\sqrt{(2N)}\sigma}\right]\right) = 1$$

$$\lim_{\sigma\to\infty} P_f = 1 - \int_{u=-\infty}^{+\infty} \left[\prod_{s=1}^{N-1} \left(\frac{1}{2}\right)\right]^2 \frac{1}{\sqrt{(2\pi N)}\sigma} \times \exp\left[-\frac{(u-N)^2}{2N\sigma^2}\right] du$$

$$\lim_{\sigma\to\infty} P_f = 1 - \left(\frac{1}{2}\right)^{2(N-1)} \int_{u=-\infty}^{+\infty} \frac{1}{\sqrt{(2\pi N)}\sigma} \times \exp\left[-\frac{(u-N)^2}{2N\sigma^2}\right] du$$

$$\lim_{\sigma\to\infty} P_f = 1 - \left(\frac{1}{2}\right)^{2(N-1)}$$

6.3 Limit for eqn. 43 as $\sigma \to \infty$

$$\lim_{\sigma\to\infty} P_f = 1 - \left[\prod_{s=1}^{N-1} \left(\frac{1}{2}\right)^{|s|} \underbrace{\sum_{v=0}^{|s|} \binom{|s|}{v}}_{2^{|s|}}\right]^2 \left(\frac{1}{2}\right)^{2(N-1)}$$

$$\lim_{\sigma\to\infty} P_f = 1 - \left(\frac{1}{2}\right)^{2(N-1)}$$