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The analytic treatment of the error probability due to clipping — — a solved problem?

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Abstract

The work describes the shortcomings of current analytical treatments of the noise and error probability caused by the clipping of a multicarrier signal. The best approach so far by Bahai et al. is shown not to describe the actual properties correctly. The shortcomings of the current approaches are outlined and simulation results of the real behavior are shown. This presentation is intended to be a starting point for a new analysis.

1. MOTIVATION

The high peak-to-average ratio (PAR) is the major drawback of multicarrier transmission. It requires for high resolution converters and high-voltage power supplies, in turn leading to high power dissipation. Without PAR reduction schemes in place, at some voltage level, the signals will be clipped. It would thus be important to know the level of noise and its frequency dependency defining the bit-error rate on every carrier. Two possible approaches had been given in the past, which are subsequently studied. Thereafter, we present some simulation result that show the real clipping distortion.

2. THE TWO APPROACHES FOR ANALYZING CLIPPING DISTORTION

In the past, the most often used approach is to see the effect of clipping as white Gaussian noise in DFT domain (see, e.g., [1]). Choosing the DFT representation with $1/\sqrt{N_f}$ in both transform equations (N_f being the block length), the noise power N will be the same and just given by the clipped portion of the Gaussian density in time domain. This means

$$N = 2\int_{\mathcal{L}_c\sqrt{P}}^{\infty}(u-\mathcal{L}_c\sqrt{P})^2\frac{1}{\sqrt{2\pi P}}e^{-\frac{u^2}{2P}}du \qquad (1)$$

with the normalized clipping level $\mathcal{L}_c = l/\sqrt{P}$, the clipping voltage l, and the signal variance (power) P. With the Gaussian i.i.d. assumption, the probability p_d of crossing the half distance between two neighboring QAM signal points can easily be computed as

$$p_d = \frac{1}{2}\,\mathrm{erfc}\left(\frac{a/2}{\sqrt{2N}}\right), \qquad (2)$$

with a denoting the minimum Euclidean distance between points. Determining the corresponding bit-error probability is straight forward.

Unfortunately, as has been correctly pointed out by Bahai et al. in [2], there are typically not so many clipping events within a DMT(OFDM) symbol that could justify a Gaussian i.i.d. assumption for the DFT domain. This kind of analysis is therefore only valid for very low clipping levels with many clips. For applications, this is an unusual case. Ignoring this, the bit-error rate computations would lead to the completely wrong interpretation that clipping noise is not an issue at all. Very low error rates are obtained. The effect is underestimated as can be seen from Fig. 1 taken from [1]. One would conclude that only a minor increase in resolution would be required.

Bahai et al. [2] realized the shortcomings for the first time and went through a lengthy derivation[1] correctly assuming the bursty impulse-noise nature of clipping. The actual results, however, did not look too realistic, either. Here, we now summarize the problems

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[1] which is not presented in detail in [2]; [7] provides the intermediate steps.

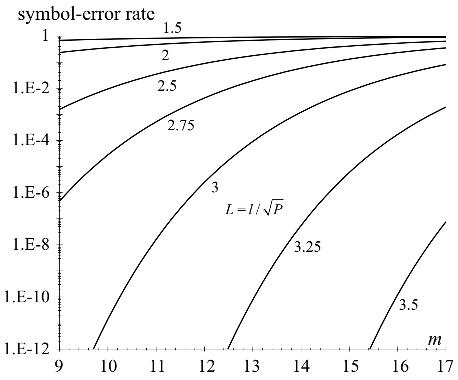

Figure 1: Symbol-error probability P_s as a function of the number of bits per carrier m, i.e., 2^m-QAM, and the clipping ratio $\mathcal{L}_c = l/\sqrt{P}$ under idealizing Gaussian i.i.d. assumptions

that we saw when going through Bahai's assumptions and derivations in detail, some of them referencing papers dating back to the 50's.

The starting point is the assumption that the clipped portion of the signal will follow a parabolic function

$$p_\tau(t) = \left(-\frac{1}{2} l m_2 t^2 + \frac{1}{8} l m_2 \tau^2\right) \cdot \mathrm{rect}\left(\frac{t}{\tau}\right) , \quad (3)$$

with

$$m_i = \begin{cases} \frac{1}{2\pi}\int \omega^i S_x(\omega) d\omega \ , & i = 2u \\ 0 \ , & i = 2u+1 \end{cases} \quad \text{for } u = 0, 1, 2, \cdots . \quad (4)$$

rect(.) denotes a rectangular window function, and τ, the random duration of the clip, forms the support of the parabolic arc. The parabolic shape of the signal's crossing above high level l follows naturally from a Taylor series expansion around the peak value. However, still the question is, if this is a realistic model, although this assumption is so appealing. In reality, most of the clips are very spiky and the actual shape depends on filter functions in place before the non-linearity.

This assumption of a parabolic shape directly leads to the spectral properties in DFT-domain, given as

$$F_k = \frac{\sqrt{N_f} m_2 T l \tau}{4\pi^2 k^2} e^{-(j2\pi k(t_0 + \frac{\tau}{2})/T)} . \quad (5)$$

$$\cdot \left(\mathrm{sinc}\left(\frac{\pi k \tau}{T}\right) - \cos\left(\frac{\pi k \tau}{T}\right)\right) , \quad (6)$$

with k is the carrier number, T the symbol (frame) duration, and t_0 the clip time, The actual noise power density spectra that we found in simulations did not follow this function. Usually occurring short single spiky noise impulses due to clipping indeed produce an almost white spectrum. The only spectral dependency of the SNR that needs to be taken into consideration is caused by any kind of filtering before the non-linearity. This filtering will influence the signal PSD (power spectral density), but since the clipping noise is almost white, the noise PSD after zero-forcing frequency equalization will increase accordingly. In other words, the SNR will, of course, drop where the signal is suppressed. This is of much stronger impact than any spectral dependency of clipping noise that one can assume. In general, there is a frequency dependency when not just investigating single clip cases, but as is shown later, it is not following (6).

The parameter that influences the shape of the spectrum is τ. It's statistics are stated in [2], as well. It's density is said to be

$$\rho_\tau(\tau) = \frac{\pi}{2}\frac{\tau}{\tau_m^2}\exp(-(\pi/4)(\tau/\tau_m)^2), \ \tau \geq 0 , \quad (7)$$

where τ_m denotes the expectation of τ. Letting λ_l be the average rate of the Poisson arrivals of clips (another assumption), $\lambda_l \tau_m = Pr\{x(t) \geq l\}$ leads to the expected value of the duration of a clip

$$\tau_m = \frac{Pr\{x(t) \geq l\}}{\lambda_l} \approx \frac{\sqrt{2\pi}}{l\sqrt{m_2}} , \quad (8)$$

The approximation is due to

$$\begin{aligned}
& \Pr\{x(t) \geq l\} = \\
= \ & \frac{1}{\sqrt{2\pi}} \int_{u=l}^{u=\infty} e^{-u^2/2} \, du \\
= \ & \frac{1}{\sqrt{2\pi}} \int_{u=l}^{u=\infty} -\frac{1}{u} \, d(e^{-u^2/2}) \\
= \ & \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-u^2/2}}{u}\right]_{u=l}^{u=\infty} - \frac{1}{\sqrt{2\pi}} \int_{u=l}^{u=\infty} \frac{e^{-u^2/2}}{u^2} \, du \\
= \ & \frac{1}{\sqrt{2\pi} \cdot l} \, e^{-l^2/2} + \frac{1}{\sqrt{2\pi}} \int_{u=l}^{u=\infty} \frac{1}{u^3} \, d(e^{-u^2/2}) \\
= \ & \frac{1}{\sqrt{2\pi} \cdot l} \, e^{-l^2/2} - \frac{1}{\sqrt{2\pi} \cdot l^3} \, e^{-l^2/2} + \cdots \\
= \ & \frac{1}{\sqrt{2\pi} \cdot l} \, e^{-l^2/2} \left\{1 - \frac{1}{l^2} + \frac{3}{l^4} - \cdots\right\} . \quad (9)
\end{aligned}$$

Together with

$$\lambda_l = \frac{\sqrt{m_2}}{2\pi}\exp(-l^2/2) , \quad (10)$$

which is also derived in [2], the approximation results when using only the first term in (9). We still have to compute

$$m_2 = \frac{1}{2\pi}\int \omega^2 S_x(\omega) d\omega , \quad (11)$$

which equals

$$m_2 = \frac{(2\pi)^2 f_0^3 S_0}{3} \tag{12}$$

for a constant PSD S_0 within a bandwidth of f_0. Bahai normalized the total power to be one, which means that $S_0 = 1/f_0$, finally leading to

$$m_2 = \frac{(2\pi)^2 f_0^2}{3} \, . \tag{13}$$

The actual distribution function in (7) and hence, the resulting density was taken from an early paper by Rice [5] from 1958, which he had called a '*conjecture*'. However the actual distribution looks like, let us have a look at the realistic value of τ_m. Let us choose a very low normalized[2] clipping level of 10 dB. This would mean a normalized clipping level of $l = 10^{10/20} = 3.16$ and let us further use the parameters of ADSL, i.e., $f_0 = 1.104$ MHz. We then obtain

$$\tau_m = \frac{\sqrt{3}}{l\sqrt{2\pi} f_0} = 0.22/f_0 \approx 0.2\,\mu\text{s} \, . \tag{14}$$

The clock rate is 2.208 MHz, meaning a sample time of 0.453 μs. The mean duration of a clip would be significantly below the duration of a sample. This is in accordance with our finding that the clipping spikes that we see are just single samples, leading to a white disturbance in DFT domain. Thus, we need not really discuss, if the assumption of the actual shape of the distribution is correct. Most of the clip events will be single samples. If we are concerned with the out-of-band extension, then the actual duration may be an issue, but this was not the content of Bahai's work and will not be treated here, either. However, one would need to study Rice's work very carefully, if the given distribution really holds. Furthermore, note that the normalized clipping level of 10 dB is quite low. Without any means of peak reduction, 14 dB should be more realistic, leading to an even shorter τ_m.

Thus, we conclude that although the assumption of the shape and length distribution of clips may come with some question marks, practically, they will not lead to a frequency dependency in the case of single clips. Multiple clips were not treated in [2]. What we will practically see are either single-sample clips with a white dependent (not i.i.d. as in [1]) disturbance in DFT domain or multiple single-sample clips with possibly some frequency effect. The second has not been studied, yet, but is visible in simulation.

There are a few further (minor) weaknesses in [2] that we mention shortly. When looking into the derivation of the amplitude of the parabolic arc exceeding level l thoroughly (see Fig. 2), it starts from a Taylor extension using only the constant and quadratic term given as

$$p_\tau(t) = \left(\frac{1}{2}x'' t^2 - \frac{1}{8}x'' \tau^2\right) \cdot \text{rect}\left(\frac{t}{\tau}\right) \, . \tag{15}$$

The quadratic term of a Taylor expansion is $\frac{1}{2!}x'' t^2$ around $t = 0$. The constant term ensures the zeros at $t = \pm\tau/2$. x'' can be approximated by $x'' \cong R''_{xx}(0) \cdot x$, R''_{xx} being the second derivative of the autocorrelation function. In [2], however, x is replaced by l, the clipping level, although x exceeds l.

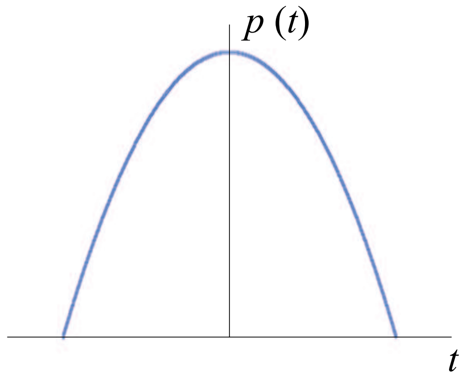

Figure 2: Parabolic arc modeling the excursion above level l

In the course of the actual derivation of the error probability, a variable substitution was made to obtain a Gaussian distribution. The corresponding standard deviation should correctly be

$$\sigma = \left(\frac{1}{\sqrt{3N_f}\pi l^2}\right)^{1/3} \, , \tag{16}$$

whereas $\sigma = \left(\frac{2}{\sqrt{3N_f}\pi l^2}\right)^{1/3}$ is given in [2] below (26).

In the course of deriving (27) of [2], the authors make use of their normalization of the total power to one. However, they do not rigorously follow the same procedure as for σ. A normalization of the power to unity means that the rms value in time domain will be unity as well (despite of a reference impedance, which can be ignored). Applying a power-preserving DFT, this would again mean that the average power in DFT domain will also be one. It will not be $1/N$ per carrier as stated in [2], although it is indeed appealing to just divide the normalized power by N to obtain the power per carrier in the discrete representation.

[2] normalized to the standard deviation

Not all derivations can be given here in detail. We will soon provide [7] on our Web page, which especially contains intermediate derivation steps of [2].

3. SOME SIMULATION RESULTS

We first show the mean PSDs resulting from clipping at two normalized clip levels. They are given in Fig. 3.

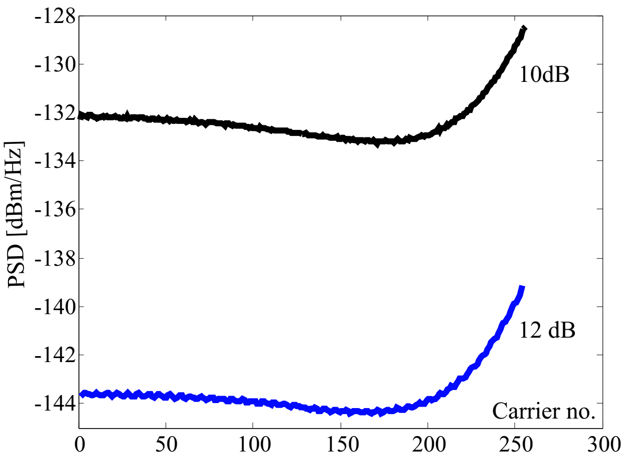

Figure 3: Noise power spectral density at normalized clipping levels of 10 and 12 dB

The decrease with frequency is due to multiple clips. If we would have plotted it only for single clips, the spectrum would just be white, except a rise at higher frequencies. The increase there is due to the filtering that we applied before clipping. The FEQ would then increase the noise which clearly can be seen. This is much stronger than any other frequency dependency. This low average noise level in the order of usual background noise could be interpreted as nothing to worry about. A three-dimensional plot in Fig. 4 of the distribution of the PSD levels outlines that this interpretation may be misleading. There are certainly some clipping events that reach up to about -70 dBm/Hz. One may compare with, e.g., the so-called ETSI A noise model specified in G.996.1, which has a maximum at -100 dBm/Hz and the background level at -140 dBm/Hz.

4. CONCLUSIONS

We discussed two possible approaches for describing the disturbance due to clipping, the Gaussian-i.i.d. approach and the single-parabolic-impulse approach by

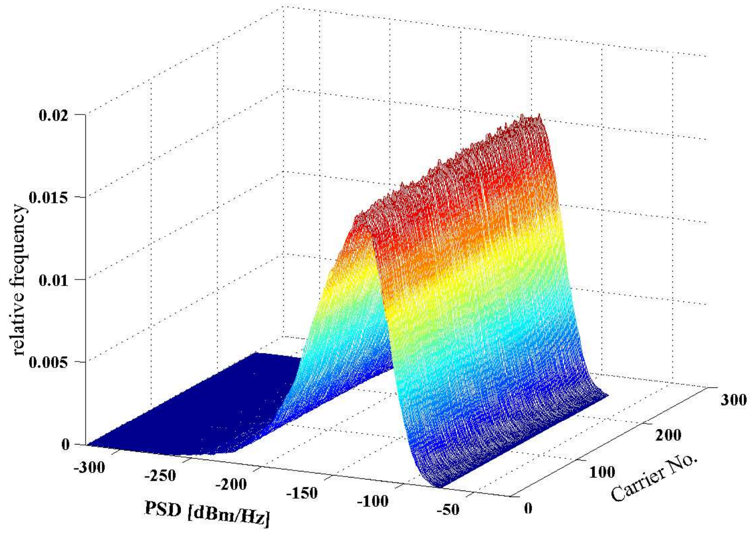

Figure 4: Histogram of the noise PSD at a normalized clipping level of 10 dB

Bahai et al.. Unfortunately, both do not yet describe the real spectral effect. This paper outlines the shortcomings, not yet a new solution. As a conference presentation, it should serve as a starting point for a new effort in this direction.

References

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