An Extension of the Levinson-Durbin Algorithm for the Inversion of Toeplitz Matrices

An extension of the well-known Levinson-Durbin algorithm for the inversion of Toeplitz matrices is given. The structure is tree-like, included in and making use of the conventional recursions of the method. Its use is restricted to the case of nonsingular leading principal submatrices.

Eine Erweiterung des Levinson-Durbin-Algorithmus zur Inversion von Toeplitz-Matrizen

Es wird eine Erweiterung des bekannten Levinson-Durbin-Algorithmus vorgestellt, welche die Inversion von Toeplitz-Matrizen ermöglicht. Hierbei dürfen keine Singularitäten in den betrachteten Untermatrizen auftreten. Das Verfahren weist eine Baumstruktur auf und verwendet lediglich die üblichen Operationen des herkömmlichen Algorithmus.

1. Introduction

Levinson's Algorithm has been introduced in 1947 [1], shortly after the work of Wiener [2], as a method for the solution of discrete least-squares estimation problems. In 1960 it was rediscovered and refined by Durbin [3].

The algorithm solves Toeplitz systems (not only symmetric or Hermitian ones as stated in [4]) that appear in several disciplines like inverse scattering, estimation, linear prediction and the decoding of Reed-Solomon codes. But its use is restricted to cases, where all leading principal submatrices are known to be nonsingular (not taking works of Pombra, Lev-Ari and Kailath [7] into consideration).

Here an extension of the algorithm is given, that not only allows to solve Toeplitz systems but invert Toeplitz matrices (under the condition mentioned).

2. The Original Levinson-Durbin Algorithm

[1] As an example for a Toeplitz system of equations the so-called 'Yule-Walker equations' are considered. These are given by

$$(1, A_{m,1}, \ldots, A_{m,m}) \underbrace{\begin{pmatrix} R_0 & R_1 & \cdot & \cdot & R_m \\ R_{-1} & R_0 & \ddots & & \cdot \\ \cdot & \ddots & \ddots & \ddots & \cdot \\ \cdot & & \ddots & \ddots & R_1 \\ R_{-m} & \cdot & \cdot & R_{-1} & R_0 \end{pmatrix}}_{R_m} = (R_m^l, 0, \ldots, 0), \quad (1)$$

where R_m^l labels the mean-square error and $\boldsymbol{R}_m$ is the $(m+1)\times(m+1)$-autocorrelation-matrix.

The recursions of the algorithm start with the shortest possible length of the vector $\boldsymbol{A}$, with the system

$$(1)\,(R_0) = (R_0^l) = (R_0^r), \quad (2)$$

bearing in mind that the shortest predictor with acceptable mean-square error has to be determined.

This means, the method starts at the upper left or even at the lower right corner of the coefficient matrix. (Remark: The Berlekamp-Massey algorithm, on the contrary, begins with the lower left or upper right corner.)

The quite reasonable procedure to increase the length of the vector and the size of the matrix is to append a zero to the right of the vector $\boldsymbol{A}$ (no length change of the predictor). Considering the system in (1), an enlargement in this way yields

$$(1, A_{m,1}, \ldots, A_{m,m}, 0)(\boldsymbol{R}_{m+1}) = (R_m^l, 0, \ldots, 0, \alpha_m). \quad (3)$$

In the case of $\alpha_m = R_{m+1} + \sum_{i=1}^{m} A_{m,i} R_{m-i+1} \neq 0$, a second solution vector that is determined in parallel by leftsided adding of zeros, is used to eliminate this component. The second solution vector, labelled $\boldsymbol{B}$, having a reverse structure, leads to

$$(0, B_{m,m}, \ldots, B_{m,1}, 1)(\boldsymbol{R}_{m+1}) = (\beta_m, 0, \ldots, 0, R_m^r) \quad (4)$$

for leftsided supplement by zero.

Elimination of the component α_m is now achieved by combining both vectors:

$$(1, A_{m+1,1}, \ldots, A_{m+1,m+1}) = \quad (5)$$
$$= (1, A_{m,1}, \ldots, A_{m,m}, 0) + K_m^\alpha (0, B_{m,m}, \ldots, B_{m,1}, 1),$$

where the factor K_m^α has to be chosen as

$$K_m^\alpha = -\alpha_m (R_m^r)^{-1}. \quad (6)$$

Correspondingly, β_m is forced to zero by

$$(B_{m+1,m+1}, \ldots, B_{m+1,1}, 1) = \quad (7)$$
$$= (0, B_{m,m}, \ldots, B_{m,1}, 1) + K_m^\beta (1, A_{m,1}, \ldots, A_{m,m}, 0),$$

if K_m^β equals

$$K_m^\beta = -\beta_m (R_m^l)^{-1}. \quad (8)$$

Hence, recursions for R_m^l and R_m^r are obtained as follows:

$$R_{m+1}^l = R_m^l - \alpha_m \beta_m (R_m^r)^{-1}, \quad (9)$$

$$R_{m+1}^r = R_m^r - \alpha_m \beta_m (R_m^l)^{-1}. \quad (10)$$

[1] This relies partly on [6].

Regarding that the initialisation yields

$$R_0^r = R_0^l = R_0, \tag{11}$$

the equality of R_m^l and R_m^r follows from (9) and (10).

Now, after explaining the original LDA, the new extension for the inversion of Toeplitz matrices is described.

3. The Extended LDA

Before proceding to the extension itself, an important property of Toeplitz matrices should be mentioned. Toeplitz matrices belong to the so-called 'persymmetric' matrices (see e.g. [4] or [5]) with a symmetry about its antidiagonal. It can be shown, that their inverse matrices are persymmetric, too. (The proof uses the fact that a matrix A is persymmetric if and only if $JAJ = A^T$ with an exchange matrix J having all ones along the cross-diagonal and zeros elsewhere.)

To come to the extended LDA, the main operations of the usual LDA are pointed out more clearly. These are:

Enlargement by adding a zero to the right:

The right side of the system of equations remains unchanged and a new component is appended righthand. Represented graphically:

(← previous right side →, new component).

Enlargement by adding a zero to the left:

The right side of the system of equations remains unchanged and a new component is appended lefthand. Represented graphically:

(new component, ← previous right side →).

This means that the previous right side either appears flushed left or flushed right. By combining with the vectors A or B of the conventional LDA, it is possible to obtain a right side that only has one component not equal to zero. By normalizing to that component one achieves one row of the inverse Toeplitz matrix. A possible procedure to obtain the whole inverse is given in form of a block diagram in Fig. 1.

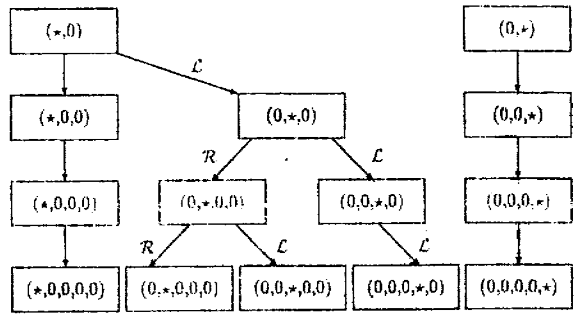

Fig. 1. Schematic diagram of the extended Levinson-Durbin-Algorithm.

The conventional LDA is to be found on the left and right of the diagram (vertical arrows). Arrows oriented to the right mark lefthand zero extension ($\mathscr{L}$) and those oriented to the left mark righthand zero extension ($\mathscr{R}$). The new components are eliminated by using the solutions of the conventional LDA on the same level.

For illustration, an example for the inversion of a 4 × 4-matrix over GF (11) is given. Pay also attention to the fact that the resulting inverse is persymmetric but not Toeplitz.

Example

$$A = \begin{pmatrix} 10 & 0 & 4 & 0 \\ 2 & 10 & 0 & 4 \\ 9 & 2 & 10 & 0 \\ 5 & 9 & 2 & 10 \end{pmatrix}$$

$$(1)\,(10) = (10)$$

$(1, 0)\begin{pmatrix} 10 & 0 \\ 2 & 10 \end{pmatrix} = (10, 0)$	$(0, 1)\begin{pmatrix} 10 & 0 \\ 2 & 10 \end{pmatrix} = (2, 10)$
$(1, 0) - \frac{0}{10}(0, 1) = (1, 0)$	$(0, 1) - \frac{2}{10}(1, 0) = (2, 1)$
$(1, 0)\begin{pmatrix} 10 & 0 \\ 2 & 10 \end{pmatrix} = (10, 0)$	$(2, 1)\begin{pmatrix} 10 & 0 \\ 2 & 10 \end{pmatrix} = (0, 10)$

$(1,0,0)\begin{pmatrix}10&0&4\\2&10&0\\9&2&10\end{pmatrix}=(10,0,4)$ $(1,0,0)-\frac{4}{10}(0,2,1)=(1,8,4)$	$(0,1,0)\begin{pmatrix}10&0&4\\2&10&0\\9&2&10\end{pmatrix}=(2,10,0)$ $(0,1,0)-\frac{2}{7}(1,8,4)=(6,5,2)$	$(0,2,1)\begin{pmatrix}10&0&4\\2&10&0\\9&2&10\end{pmatrix}=(2,0,10)$ $(0,2,1)-\frac{2}{10}(1,0,0)=(2,2,1)$
$(1,8,4)\begin{pmatrix}10&0&4\\2&10&0\\9&2&10\end{pmatrix}=(7,0,0)$	$(6,5,2)\begin{pmatrix}10&0&4\\2&10&0\\9&2&10\end{pmatrix}=(0,10,0)$	$(2,2,1)\begin{pmatrix}10&0&4\\2&10&0\\9&2&10\end{pmatrix}=(0,0,7)$

$(1,8,4,0)\,A=(7,0,0,10)$ $(1,8,4,0)-\frac{10}{7}(0,2,2,1)=(1,2,9,8)$	$(6,5,2,0)\,A=(0,10,0,9)$ $(6,5,2,0)-\frac{9}{3}(4,1,7,1)=(5,2,3,8)$
$(1,2,9,8)\,A=(3,0,0,0)$	$(5,2,3,8)\,A=(0,10,0,0)$

$(0,6,5,2)\,A=(1,0,10,0)$ $(0,6,5,2)-\frac{1}{3}(1,2,9,8)=(7,9,2,3)$	$(0,2,2,1)\,A=(5,0,0,7)$ $(0,2,2,1)-\frac{5}{7}(1,8,4,0)=(4,1,7,1)$
$(7,9,2,3)\,A=(0,0,10,0)$	$(4,1,7,1)\,A=(0,0,0,3)$

$$\Rightarrow A^{-1}=\begin{pmatrix}1/3&2/3&9/3&8/3\\5/10&2/10&3/10&8/10\\7/10&9/10&2/10&3/10\\4/3&1/3&7/3&1/3\end{pmatrix}=\begin{pmatrix}4&8&3&10\\6&9&8&3\\4&2&9&8\\5&4&6&4\end{pmatrix}$$

(Received August 27, 1989.)

Dr.-Ing. Werner Henkel
Technische Hochschule Darmstadt
Merckstraße, D-6100 Darmstadt
now with
Forschungsinstitut der Deutschen Bundespost
beim Fernmeldetechnischen Zentralamt
Am Kavalleriesand 3
D-6100 Darmstadt

References

[1] Levinson, N.: The Wiener RMS (root mean square) error criterion in filter design and prediction. J. of Math. and Phys. **25** (1947), 261–278.

[2] Wiener, N.: Extrapolation, interpolation and smoothing and stationary time series, with engineering applications. New York: Technology Press, 1949.

[3] Durbin, J.: The fitting of time-series models. Rev. Int. Statist. Inst. **28** (1960), 233–244.

[4] Blahut, R. E.: Fast alforithms for digital signal processing. 2nd printing. Reading: Addison-Wesley, 1987.

[5] Zohar, S.: Toeplitz matrix inversion: The algorithm of W. F. Trench. J. Assoc. Comp. Mach. **16** (1969), 592–601.

[6] Kailath, T.: A view of three decades of linear filtering theory. IEEE Trans. IT-**20** (1974), 146–181.

[7] Pombra, S.; Lev-Ari, H.; Kailath, T.: Levinson and Schur algorithms for Toeplitz matrices with singular minors. Proc. ICASSP-**88** (1988), 1643–1646.

[8] Robinson, E. A.: Multichannel time-series analysis with digital computer programs. San Francisco: Holden-Day, 1967.

[9] Henkel, W.: Zur Decodierung algebraischer Blockcodes über komplexen Alphabeten. Fortschrittsberichte, Reihe 10, Nr. 109. Düsseldorf: VDI-Verlag, 1989.