Hierarchical Modulation to Support Low-Density Parity-Check Code Design?

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***Abstract*— We study variable node degree distributions resulting from a linear programming approach for designing a low-density parity-check (LDPC) code optimized for an irregular modulation alphabet. We opt for an irregular LDPC code and make use of the irregularities of the channel in the sense of different modulation classes originating especially from different distances in a hierarchical modulation signal set. Our aim is to investigate if differences in equivalent binary channels as given by hierarchical modulation signal sets would in any way support resulting variable node degree distributions.**

I. Introduction and Motivation

Low-density parity-check (LDPC) codes, first introduced by Robert Gallager in his PhD thesis [1], have been shown to perform very close to capacity, as described by the Shannon capacity formula. These results have motivated further research on LDPC codes for different channels and noise scenarios.

For this research work, we focus on designing irregular LDPC codes which, as presented in the literature, have been shown to perform better in comparison to the regular ones. In [2], the authors were interested in providing unequal error protection (UEP) for data frames and considered intentional non-uniformities in the error probabilities for different sets of bits to protect some more than others. For our approach, we introduce non-uniformities by using a hierarchical modulation which results in different error probabilities for the different sets of bits inside a symbol. The level of non-uniformity to be offered can be flexibly varied with the help of the constellation parameter given by the ratio of the inter-symbol distances describing the constellation. Apart from the natural protection resulting from the hierarchical modulation, in here, we are not aiming at overall UEP capabilities.

Further on, by keeping the check node degree distribution fixed, we optimize the variable node degree distribution after dividing it into sub-degree distributions corresponding to the different modulation classes. The optimization is done using density evolution and a linear programming technique based on a rate maximization criterion.

The paper is organized as follows. Section II introduces the concept of hierarchical modulation along with the investigation of the different error probabilities for the bits inside a 16-QAM hierarchical modulated symbol chosen as an example. Section III presents the LDPC code design for a higher order constellation (HOC) as well as the system model. Section IV represents the main part of this paper describing the optimization strategy in general as well as the one employed for the irregular hierarchical modulation. In this section, the LP algorithm as well as details regarding the code construction are explained. In Section V, the simulation results are presented and analyzed, followed by conclusions in Section VI.

II. Hierarchical Modulation

This section is devoted to introducing the concept of hierarchical modulation as well as presenting the motivation behind using it together with irregular LDPC codes.

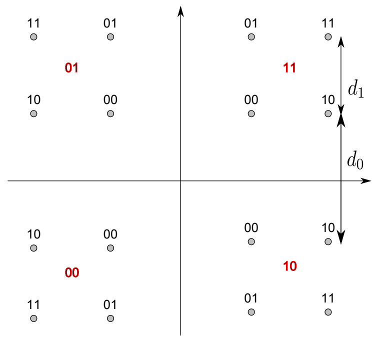

Fig. 1. 16-QAM hierarchical constellation with Gray-coded bit mapping

From Fig. 1, we observe that the 16-QAM hierarchical constellation is formed by an outer QPSK scheme which is addressed by the two most significant bits (MSBs) of the symbol and an inner QPSK scheme described by the remaining least significant bits of the symbol (LSBs). Therefore, we obtain two equivalent channels (two modulation classes). The non-uniformity of the symbols in the hierarchical constellation results in error probabilities which are considerably different[1] for the individual bits inside the symbol. Due to this, hierarchical modulation is of interest when UEP inside a modulation symbol is desired. The constellation parameter, $\alpha = d_0/d_1$, dictates how much more the first channel is protected against errors than the second one.

Using the exact formulas derived in [3] for the channels' bit-error probabilities for the case of a Gray-coded hierarchical 16-QAM constellation, we obtain

$$P_{b,c_1} = \left(\frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_s}{N_0}}\frac{d_0}{2}\right) + \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_s}{N_0}}\left(\frac{d_0}{2}+d_1\right)\right)\right)$$

[1]The uniform constellation, characterized by $\alpha = 1$, also provides some inherent irregularities for the bits in the modulation symbol, however, hierarchical values of α allow for a greater distinction in the equivalent binary channels.

$$P_{b,c_2} = \frac{1}{2}\left(\text{erfc}\left(\sqrt{\frac{E_s}{N_0}}\frac{d_1}{2}\right) + \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_s}{N_0}}\left(d_0 + \frac{d_1}{2}\right)\right) - \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_s}{N_0}}\left(d_0 + \frac{3d_1}{2}\right)\right)\right) , \quad (1)$$

where "erfc" denotes the complementary error function and E_s/N_0 represents the signal-to-noise ratio with respect to the symbol energy[2]. The bit-error probability of the first channel P_{b,c_1} is also the bit-error probability of the first two bits (P_{b,b_0} and P_{b,b_1}) in the modulation symbol while P_{b,c_2} corresponds to the remaining two bits (P_{b,b_2} and P_{b,b_3}).

The individual noise variances for the different modulation classes can be computed from the corresponding bit-error probability using an equivalent BPSK description of the channels as in [4]. Thus, the error probability of the resulting channels and the noise variances are linked by

$$\sigma_{c_j}^2 = \frac{1}{2\left(\text{erfc}^{-1}\left(2P_{b,c_j}\right)\right)^2} . \quad (2)$$

Our aim is to investigate if the non-uniformity introduced by the different error probabilities of the bits, as a result of using hierarchical modulation, will lead to less irregular variable node degree distributions for the resulting LDPC codes or other trends become visible.

III. Code Design and System Model

Making use of a hierarchical modulation results in different modulation classes described by different noise variances. Consequently, the standard LDPC code design, which assumes that all the bits are transmitted over one channel and are equally protected against errors, does not apply. A new design which meets the properties of a hierarchical HOC is therefore required and was introduced in [4]. The assumptions made for the two designs, for the 16-QAM case, are presented in Fig. 2.

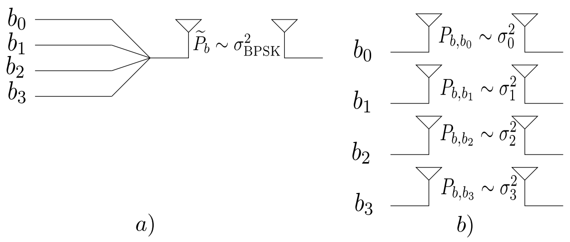

Fig. 2. Assumptions made for a) Standard code design and b) HOC design

In the above, $\sigma_0^2 = \sigma_1^2 = \sigma_{c_1}^2$, $\sigma_2^2 = \sigma_3^2 = \sigma_{c_2}^2$ while σ^2_{BPSK} represents the noise variance of a channel that would be described by the average bit-error probability of the HOC scenario and provides a fair way to compare the two designs.

After applying the resulting LDPC code, every codeword bit is assigned to one of the modulation classes available (M_j with $j = 1, \ldots, N_s$). Each modulation class is described by a different bit-error ratio.

[2]The relation between the symbol energy and the energy of an information bit in the case of a coded system with modulation rate $R_m = \log_2(M)$ and code rate R_c is $E_s = R_c \cdot R_m \cdot E_b$.

A. Notations

LDPC codes are linear block codes described by a sparse parity-check matrix $\mathbf{H}$ with dimensions $(N-K) \times N$ such that N represents the length of the codeword, K the length of the information word and the design rate of the code is given by $R = K/N$. LDPC codes are usually represented using a bipartite (Tanner) graph which is constructed using two categories of nodes, variable and check nodes, mapping to the elements of the codeword and to the parity check constraints, respectively. The two types of nodes are connected by edges which correspond to the non-zero entries of the $\mathbf{H}$ matrix. We can distinguish two types of graphs, regular and irregular. Regular LDPC codes have the property that inside each class of nodes all the nodes have the same degree. If the degrees of the variable/check nodes vary among the classes of nodes, we refer to them as irregular LDPC codes. The irregular variable and check node degree distributions are defined using polynomials

$$\lambda(x) = \sum_{i=2}^{d_{v_{max}}} \lambda_i x^{i-1} \text{ and } \rho(x) = \sum_{i=2}^{d_{c_{max}}} \rho_i x^{i-1} , \quad (3)$$

where $d_{v_{max}}$ and $d_{c_{max}}$ represent the maximum variable and check node degrees while λ_i and ρ_i give the proportion of edges connected to variable and check nodes of degree i, respectively [5].

Using a higher order constellation (HOC), we obtain N_s modulation classes. The vector $\boldsymbol{\beta} = [\beta_1, \cdots, \beta_{N_s}]$ describes the proportion of bits assigned to each modulation class.

$\lambda(x)$, with the corresponding vector $\boldsymbol{\lambda}$, contains the overall variable node degree distribution coefficients. Accordingly, $\lambda_{M_j,i}$ represents the proportion of edges connected to variable nodes of degree i that belong to the modulation class M_j. We also define $\boldsymbol{\lambda}_{M_j}$ to be the vector characterizing the sub-degree distributions such that $\boldsymbol{\lambda}_{M_j} = \left[\lambda_{M_j,2}, \ldots, \lambda_{M_j,d_{v_{max}}}\right]^T$ and $\boldsymbol{\lambda} = \left[\boldsymbol{\lambda}_{M_1}, \cdots, \boldsymbol{\lambda}_{M_{N_s}}\right]^T$. The check node degree distribution is described by the vector $\boldsymbol{\rho} = \left[\rho_2, \ldots, \rho_{d_{c_{max}}}\right]^T$ while E is the total number of edges in the graph. Using the above, we have

$$N_{M_j} = \beta_j \cdot N , \quad (4)$$

where N is the total number of variable nodes and N_{M_j} the number of variable nodes associated with the modulation class M_j. Further on, N_{M_j} and N can be obtained also in terms of $\boldsymbol{\lambda}_{M_j}$ and $\boldsymbol{\lambda}$

$$\sum_{i=2}^{d_{v_{max}}} \frac{\lambda_{M_j,i}}{i} = \frac{N_{M_j}}{E} \text{ and } \sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \frac{\lambda_{M_j,i}}{i} = \frac{N}{E} . \quad (5)$$

IV. Irregularities in the Modulation in LDPC Codes

A. General Description

LDPC codes are decoded using belief propagation (BP), a message-passing decoding algorithm, during which messages, considered to be independent random variables, are exchanged iteratively along the edges of the graph. In [6], it was shown that the messages at the input of the variable and check nodes can be computed as mutual information using density evolution (DE) and a Gaussian approximation. DE is an algorithm used to predict the decoding performance by analyzing the distribution of the messages transmitted under message-passing decoding [7].

For the standard LDPC code design, the mutual information messages from a check to a variable node (x_{cv}) and from a variable to a check node (x_{vc}) are

$$x_{cv}^{(l-1)} = 1 - \sum_{j=2}^{d_{c_{max}}} \rho_j J\left((j-1)\,J^{-1}\left(1 - x_{vc}^{(l-1)}\right)\right) \quad (6)$$

$$x_{vc}^{(l)} = \sum_{i=2}^{d_{v_{max}}} \lambda_i J\left(\frac{2}{\sigma^2} + (i-1)\,J^{-1}\left(x_{cv}^{(l-1)}\right)\right), \quad (7)$$

where l denotes the iteration number and $J(\cdot)$ computes the mutual information as a function of the mean, $x = J(m)$ using

$$J(m) = 1 - \frac{1}{\sqrt{4\pi m}} \int_{\mathbb{R}} \log_2\left(1 + e^{-z}\right) \cdot e^{-\frac{(z-m)^2}{4m}} dz\ , \quad (8)$$

which holds for $z \sim \mathcal{N}(m, 2m)$, a consistent Gaussian random variable. From (6) and (7), DE can be summarized as

$$x_{vc}^{(l)} = F\left(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\sigma}^2, x_{vc}^{(l-1)}\right)\ . \quad (9)$$

In order to ensure that DE allows for predicting the decoding performance, some constraints have to be met. To begin with, the convergence condition guarantees that DE converges given that the mutual information increases after every iteration

$$x_{vc}^{(l)} > x_{vc}^{(l-1)} \ \text{ or } \ F\left(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\sigma}^2, x_{vc}^{(l-1)}\right) > x_{vc}^{(l-1)}\ . \quad (10)$$

Another condition that has to be fulfilled to ensure that DE converges for a mutual information close to one is the stability condition [5]. This condition represents an upper bound requirement on the number of degree-2 variable nodes

$$\frac{1}{\lambda'(0)\rho'(1)} > e^{-r} = \int_{\mathbb{R}} P_0(x)\, e^{-\frac{x}{2}} dx \overset{\text{AWGN}}{\rightarrow} e^{-\frac{1}{2\sigma^2}}\ , \quad (11)$$

where $P_0(x)$ represents the message density associated with the received values while $\lambda'(x)$ and $\rho'(x)$ are the derivatives of the degree polynomials.

B. Optimization of the Degree Distribution for HOC

With the previous explanations of our code design for HOC, we can adapt the standard DE mutual information updates to account for the resulting modulation classes. This is accomplished by splitting the variable node degree distribution into sub-degree distributions mapping to the different N_s noise variances.

We notice that only (7), the update from the variable to the check nodes, has to be adapted since the check node degree distribution is kept constant. In [8] the term describing this procedure is "multi-edge-type" construction

$$x_{vc}^{(l)} = \sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \lambda_{M_j,i} J\left(\frac{2}{\sigma_j^2} + (i-1)\,J^{-1}\left(x_{cv}^{(l-1)}\right)\right). \quad (12)$$

The stability condition (11) has to be modified as well to illustrate the different noise variances in our scheme. As a result, we make use of the proportion vector β and obtain the approximation, as presented in [4]

$$e^{-r} = \int_{\mathbb{R}} \sum_{j=1}^{N_s} \beta_j P_{0,j}(x)\, e^{-\frac{x}{2}} dx = \sum_{j=1}^{N_s} \beta_j e^{-\frac{1}{2\sigma_j^2}}\ . \quad (13)$$

A new constraint arises due to the splitting of the variable node degree distribution and requires the sub-degree distributions to function as an overall distribution

$$\sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \lambda_{M_j,i} = 1\ . \quad (14)$$

Having characterized DE for our system, we are left with optimizing the variable node degree distribution. Our optimization strategy is based on maximizing the code rate for a given E_s/N_0. This is equivalent to maximizing the denominator of the fraction from the expression below

$$R = 1 - \frac{\sum_{k=2}^{d_{c_{max}}} \frac{\rho_k}{k}}{\sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \frac{\lambda_{M_j,i}}{i}}\ . \quad (15)$$

C. Linear Programming (LP) Optimization Algorithm

In this section, we introduce the linear programming algorithm which will deliver the optimized variable node degree distribution as well as the maximized code rate. The LP algorithm follows the steps presented in [9], where this optimization strategy was first introduced. The routine requires the check node degree distribution vector $\boldsymbol{\rho}$, the constellation parameter α, the maximum variable node degree $d_{v_{max}}$, the SNR E_s/N_0, and the proportion vector $\boldsymbol{\beta}$.

Linear program

Optimize

$$\max_{\boldsymbol{\lambda}} \sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \frac{\lambda_{M_j,i}}{i}\ , \quad (16)$$

subject to

1) Proportion distribution constraints
   a) From (14) $\sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \lambda_{M_j,i} = 1$
   b) From (4) and (5), $\forall j \in \{1, \cdots, N_s - 1\}$

$$\sum_{i=2}^{d_{v_{max}}} \frac{\lambda_{M_j,i}}{i} - \beta_j \sum_{j=1}^{N_s} \sum_{i=2}^{d_{v_{max}}} \frac{\lambda_{M_j,i}}{i} = 0 \quad (17)$$

2) Convergence constraint, see (10)

$$F\left(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\sigma}^2, x_{vc}^{(l-1)}\right) > x_{vc}^{(l-1)} \quad (18)$$

3) Stability condition, see (11) and (13)

$$\sum_{j=1}^{N_s} \lambda_{M_j,2} < \left[\sum_{k=2}^{d_{c_{max}}} (k-1)\rho_k \cdot \sum_{j=1}^{N_s} \beta_j e^{-\frac{1}{2\sigma_j^2}}\right]^{-1} \quad (19)$$

D. Code Construction

Once the optimized variable node degree distribution is obtained, the next step is to construct the $\mathbf{H}$ matrix using the Progressive Edge-Growth (PEG) algorithm. The PEG algorithm maximizes the local girth of a variable node with every newly placed edge and stands out due to its low complexity accompanied by good performance especially for the case when the degree distributions are optimized using density evolution [10].

After constructing the $\mathbf{H}$ matrix, each individual variable node is assigned to one of the two channels available to ensure proper noise level during decoding and proper LLR computation. The assignment procedure makes use of the information provided by the $\mathbf{H}$ matrix (the degree of each variable node obtained as the sum of its elements along the first dimension) in combination with the information provided by the optimized sub-degree distributions (the number of variable nodes for the degrees available in each modulation class). We start from the better protected channel and assign, for each degree at a time, the required number of variable nodes as the first positions in the $\mathbf{H}$ matrix profile which map to this degree. After each round of assigning the variable nodes, the entries corresponding to the already assigned variable nodes are zeroed such that only the entries with the variable nodes not assigned yet are left. This process is accompanied by flagging the assigned positions in a vector of length N, with 1 if it corresponds to the first channel and 2 if it is to be sent on the second channel.

V. Simulation Results and Analysis

In this section, we present the simulation results obtained for a 16-QAM hierarchical constellation with Gray-coded bit mapping. For this scenario, we obtain two modulation classes, $N_s = 2$, while the proportion of bits assigned to the two modulation classes is $\beta = [0.5, 0.5]$. The associated noise variances are found using (1) and (2). Other simulation parameters are, $d_{v_{max}} = 30$, $N = 4096$, 50 decoding iterations, and $\rho(x) = 0.00749x^8 + 0.99101x^9 + 0.00150x^{10}$. This $\rho(x)$ was obtained using numerical optimization in [5] and represents the optimum check node degree distribution for a code with rate $R = 1/2$ and $d_{v_{max}} = 30$.

Using the LP routine, we obtain the optimized variable node degree distributions for different values of α at different operating points. Table I show the degree distributions obtained for a design SNR of 5 dB and 6 dB. Considering that the check node degree distribution used is optimum for codes with $R = 1/2$ ($\rho(x)$ was not further optimized according to the maximized rate), these operating points were chosen such that the rates delivered are close to $1/2$.

The variable node degree distributions obtained for the traditional ($\alpha = 1$) and for the hierarchical cases ($\alpha > 1$) did not confirm our understanding that, for the latter scenarios, the non-uniformities added by having different bit-error probabilities inside a symbol might support the LDPC code and result in less irregular degree distributions. However, we can observe other trends. To begin with, the results support the fact that for hierarchical α's the degree distributions tend to concentrate around the same degrees. Another noticeable aspect is that the sub-degree distribution mapping to the second modulation class is more irregular than the one characterizing the first modulation class.

TABLE I

Optimum sub-degree distributions

α	$\boldsymbol{\lambda}_{M_1}(x)$	$\boldsymbol{\lambda}_{M_2}(x)$
$E_s/N_0 = 5$ dB		
$\alpha = 1$	$\lambda_{M_1,2} = 0.0832$ $\lambda_{M_1,3} = 0.1559$ $\lambda_{M_1,9} = 0.0361$ $\lambda_{M_1,30} = 0.3113$	$\lambda_{M_2,2} = 0.1234$ $\lambda_{M_2,4} = 0.1007$ $\lambda_{M_2,5} = 0.0020$ $\lambda_{M_2,8} = 0.1570$ $\lambda_{M_2,30} = 0.0304$
$\alpha = \sqrt[4]{2}$	$\lambda_{M_1,2} = 0.0881$ $\lambda_{M_1,3} = 0.1598$ $\lambda_{M_1,30} = 0.3226$	$\lambda_{M_2,2} = 0.1195$ $\lambda_{M_2,3} = 0.0046$ $\lambda_{M_2,4} = 0.0979$ $\lambda_{M_2,8} = 0.1593$ $\lambda_{M_2,9} = 0.0103$ $\lambda_{M_2,30} = 0.0380$
$\alpha = \sqrt{2}$	$\lambda_{M_1,2} = 0.0915$ $\lambda_{M_1,3} = 0.1548$ $\lambda_{M_1,30} = 0.3168$	$\lambda_{M_2,2} = 0.1168$ $\lambda_{M_2,3} = 0.0132$ $\lambda_{M_2,4} = 0.0935$ $\lambda_{M_2,8} = 0.1525$ $\lambda_{M_2,9} = 0.0082$ $\lambda_{M_2,30} = 0.0526$
$\alpha = 2$	$\lambda_{M_1,2} = 0.0958$ $\lambda_{M_1,3} = 0.1448$ $\lambda_{M_1,30} = 0.3211$	$\lambda_{M_2,2} = 0.1132$ $\lambda_{M_2,3} = 0.0256$ $\lambda_{M_2,4} = 0.0856$ $\lambda_{M_2,8} = 0.1395$ $\lambda_{M_2,9} = 0.0060$ $\lambda_{M_2,30} = 0.0683$
$E_s/N_0 = 6$ dB		
$\alpha = 1$	$\lambda_{M_1,2} = 0.0927$ $\lambda_{M_1,3} = 0.2163$ $\lambda_{M_1,10} = 0.0174$ $\lambda_{M_1,30} = 0.0326$	$\lambda_{M_2,2} = 0.1349$ $\lambda_{M_2,4} = 0.0318$ $\lambda_{M_2,5} = 0.1096$ $\lambda_{M_2,9} = 0.1515$ $\lambda_{M_2,30} = 0.2131$
$\alpha = \sqrt[4]{2}$	$\lambda_{M_1,2} = 0.0955$ $\lambda_{M_1,3} = 0.2010$ $\lambda_{M_1,30} = 0.0450$	$\lambda_{M_2,2} = 0.1329$ $\lambda_{M_2,4} = 0.1056$ $\lambda_{M_2,5} = 0.0037$ $\lambda_{M_2,8} = 0.1680$ $\lambda_{M_2,30} = 0.1944$
$\alpha = \sqrt{2}$	$\lambda_{M_1,2} = 0.0990$ $\lambda_{M_1,3} = 0.1918$ $\lambda_{M_1,4} = 0.0109$ $\lambda_{M_1,30} = 0.1242$	$\lambda_{M_2,2} = 0.1297$ $\lambda_{M_2,3} = 0.0082$ $\lambda_{M_2,4} = 0.1066$ $\lambda_{M_2,8} = 0.1133$ $\lambda_{M_2,9} = 0.0607$ $\lambda_{M_2,30} = 0.1553$
$\alpha = 2$	$\lambda_{M_1,2} = 0.1046$ $\lambda_{M_1,3} = 0.1776$ $\lambda_{M_1,30} = 0.1865$	$\lambda_{M_2,2} = 0.1233$ $\lambda_{M_2,3} = 0.0290$ $\lambda_{M_2,4} = 0.0912$ $\lambda_{M_2,8} = 0.1373$ $\lambda_{M_2,9} = 0.0186$ $\lambda_{M_2,30} = 0.1318$

To be able to better assess the contribution of the individual degrees in each modulation class, we make use of the bar chart presented in Figure 3 for the 5 dB case. We can observe that, when different hierarchical α's share the same degrees, their contributions seem to follow a trend, however not the same among different degrees and modulation classes. For the first modulation class, the coefficients for degree 30 are not strictly increasing or decreasing, however this may be due to the restriction imposed by the degree distribution, i.e., the sum of the coefficients should be 1. Another observation is that the uniform case with $\alpha = 1$ is only sometimes part of the trend. The data provided in Table I for the 6 dB case, shares some of the previous characteristics, however, the overall tendency is not as strong. Even though the degree proportions are not the same, the contributions of degree 2, degree 3, and degree 9 variable nodes for the hierarchical cases, in both modulation classes, follow the same trend as the one for the 5 dB case. For the second modulation class, the contributions of degree 30 variable nodes also show a trend, however, opposite to the one present in the 5 dB case, while the degrees 4 and 8 do not present a clear trend. The difference between the two scenarios can be motivated by the code rate, which, for the 6 dB scenario is much further away from $1/2$ than the one for 5 dB.

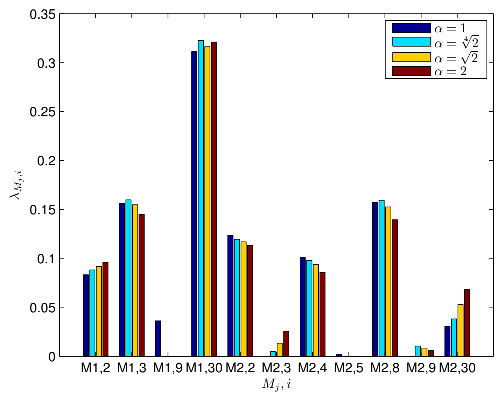

Fig. 3. Degree contribution for each modulation class at E_s/N_0 = 5 dB

From the results presented in Table II, we observe that, as expected, increasing E_s/N_0 results in higher code rates, while employing a hierarchical constellation results in lower rates when compared to the traditional case at the same E_s/N_0. One exception from this is the 5 dB case with $\alpha = \sqrt[4]{2}$ for which the rate is actually slightly higher than the one for $\alpha = 1$ and with $\alpha = \sqrt{2}$, where the rate is the same as for the case with $\alpha = 1$. We can also observe that the difference in code rates between the traditional and the non-uniform modulation increases with increasing SNR, and, evaluated at the same SNR, with increasing α.

TABLE II
MAXIMIZED CODE RATE

@ E_s/N_0	α	R
5 dB	$\alpha = 1$	0.4848
	$\alpha = \sqrt[4]{2}$	0.4855
	$\alpha = \sqrt{2}$	0.4848
	$\alpha = 2$	0.4799
6 dB	$\alpha = 1$	0.5416
	$\alpha = \sqrt[4]{2}$	0.5407
	$\alpha = \sqrt{2}$	0.5380
	$\alpha = 2$	0.5278

Figure 4 presents the bit-error ratio performance for the 5 dB operating point. Overall, the performance curves are close

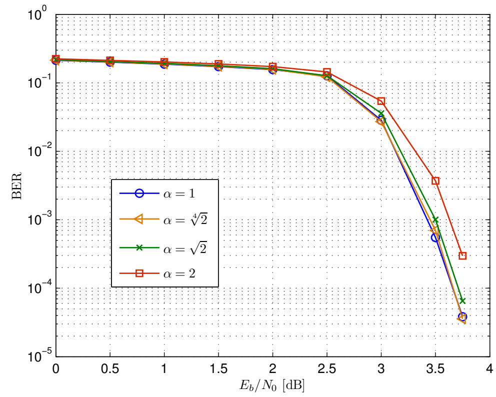

Fig. 4. Bit-error ratio performance - E_s/N_0 = 5 dB

to each other, support the characteristics already underlined by the rate performance indicator, and for the same BER requirement, the difference in E_b/N_0 for the different α's is not large. Consequently the curve associated with $\alpha = \sqrt[4]{2}$, which resulted in a slightly better rate than the traditional case with $\alpha = 1$, shows indeed a very close performance to it. Furthermore, the performance for the $\alpha = \sqrt{2}$ scenario is very similar to the $\alpha = 1$ case which again sustains the equal rate delivered by the LP.

VI. Conclusions

In this paper, we investigate if and how an unequal error protection modulation can support the design of LDPC codes. This work can easily be extended to UEP applications following a similar multi-edge-type approach. For the average performance studied in here, we have to conclude that hierarchical modulation does not significantly support the LDPC design and the average performance typically degrades due to the non-dense packing of a hierarchical signal set. However, a weak irregularity in the signal set may actually lead to a slight performance improvement. Hierarchical signal sets, i.e., unequal error protecting modulation may still be suitable to support UEP code designs.

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